PERIODIC ORBITS ABOUT AN OBLATE SPHEROID

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§ 1. *Introduction.*

The orbit of a particle about an oblate spheroid is not in general closed geometrically. The motion of the particle is not, therefore, in general, periodic from a geometric point of view. But if we consider the orbits as described by the particle in a revolving meridian plane which passes constantly through the particle several classes of closed orbits can be found in which the motion is periodic. The failure of these orbits to close in space arises from the incommensurability of the period of rotation of the line of nodes with the period of motion in the revolving plane. When these periods happen to be commensurable the orbits are closed in space and the motion is therefore periodic, though the period may be very great. Indeed, it seems that most of the difficulty in giving mathematical expressions for the orbits about an oblate spheroid rests upon the question of incommensurability of periods. The difficulty arising from the motion of the node can be overcome by the use of the revolving plane, but other incommensurabilities, such as that introduced by the eccentricity, can not be eliminated in this manner.

Orbits closed in the revolving plane are considered most conveniently in two general classes: (1) Those which reënter after one revolution, (2) those which reënter after many revolutions. The existence of both classes is established in this paper and convenient methods for constructing the solutions are given. Orbits which reënter after the first revolution are naturally the simpler and will be considered in the first part of the paper. Those lying in the equatorial plane of the spheroid become straight lines in the revolving plane, and it is shown that within the realm of convergence of the series employed all orbits in the equatorial plane are periodic. When the orbits do not lie in the equatorial plane there exists one, and only one, orbit for any arbitrarily assigned values of the inclination and the mean distance. These orbits reduce to circles with the vanishing of the oblateness of the spheroid.

In considering orbits which reënter only after many revolutions the differential equations are found to be very complex, and one would despair of proving the existence of these orbits by the ordinary direct computation of the necessary coefficients. However, a proof of their existence and a method for the construction of the solutions are given by the aid of certain theorems, which are here established, on the character of the solutions of non-homogeneous linear differential equations with periodic coefficients.

These periodic orbits of many revolutions involve five constants, four of which are entirely arbitrary, and the fifth subject only to the limitation that it shall satisfy certain commensurability conditions. One constant, only, is missing for a complete integration of the differential equations. These orbits are all symmetric with respect to the equatorial plane.

§ 2. *The differential equations.*

The differential equations of motion of a particle about an oblate spheroid are *

$$
(1) \qquad \begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{k^2Mx}{R^3}\left[1 + \tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2 + \cdots\right] = \frac{\partial V}{\partial x},\\
\frac{d^2y}{dt^2} &= -\frac{k^2My}{R^3}\left[1 + \tfrac{3}{10}b^2\,\frac{x^2+y^2-4z^2}{R^4}\,\mu^2 + \cdots\right] = \frac{\partial V}{\partial y},\\
\frac{d^2z}{dt^2} &= -\frac{k^2Mz}{R^3}\left[1 + \tfrac{3}{10}b^2\,\frac{3(x^2+y^2)-2z^2}{R^4}\,\mu^2 + \cdots\right] = \frac{\partial V}{\partial z}.
\end{aligned}
$$

The symbols employed are defined as follows: x, y, z are rectangular coördinates, the origin being at the center of the spheroid and the xy-plane the plane of the equator, k is the Gaussian constant, M is the mass of the spheroid, b is the polar radius of the spheroid, μ is the eccentricity of the spheroid,

$$R = \sqrt{x^2+y^2+z^2}, \qquad V = \frac{k^2M}{R}\left[1 + \frac{b^2}{10}\,\frac{x^2+y^2-2z^2}{R^4}\,\mu^2 + \cdots\right].$$

Since

$$\frac{1}{x}\frac{\partial V}{\partial x} = \frac{1}{y}\frac{\partial V}{\partial y},$$

we obtain one integral of areas, namely,

$$(2) \qquad x\frac{dy}{dt} - y\frac{dx}{dt} = c_1.$$

That is, the projection of the area described by the radius vector upon the equatorial plane is proportional to the time. We have also the vis viva integral

$$(3) \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2V + c_2.$$

For further integration we are compelled to resort to series. Poincaré has shown in his *Les méthodes nouvelles de la mécanique céleste* that if certain

* MOULTON, *Celestial Mechanics*, p. 113.

conditions are fulfilled it is possible to obtain periodic orbits represented by power series in a parameter when periodic orbits are known for zero values of this parameter. These conditions are at least partially fulfilled in the present problem, for the right members of the differential equations are analytic in x, y, z and the parameter μ. Furthermore, for $\mu = 0$, the equations reduce to the ordinary two-body problem for which periodic solutions are known. It is our purpose to show that the remaining necessary conditions also are fulfilled and that periodic solutions persist for values of $\mu \neq 0$. These periodic series are very satisfactory, for the general character of the orbits represented by them is easily obtained. The solutions thus found are rigorous, but they are not general, their existence depending upon special initial conditions.

It will be advantageous to transform the differential equations to cylindrical coördinates by the substitutions

$$(4)\qquad \begin{aligned} x &= ar\cos v, & k^2 M &= n^2 a^3, \\ y &= ar\sin v, & c_1 &= ck\sqrt{Ma}, \\ z &= aq, & nt &= \tau, \\ R &= a\sqrt{r^2 + q^2}, & \frac{3}{10}\frac{b^2}{a^2} &= \theta_1^2. \end{aligned}$$

After these substitutions equations (1) become

$$(5)\qquad \begin{aligned} &(a) \quad r'' - rv'^2 = \frac{-r}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{r^2 - 4q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right], \\ &(b) \quad rv'' + 2r'v' = 0, \\ &(c) \quad q'' = \frac{-q}{(r^2+q^2)^{\frac{3}{2}}}\left[1 + \frac{3r^2 - 2q^2}{(r^2+q^2)^2}\theta_1^2\mu^2 + \cdots\right], \end{aligned}$$

the accents denoting the derivatives with respect to τ.

The integral of (b) is $r^2 v' = c$, by means of which v' can be eliminated from the first of these equations. After the elimination the equations take the form

$$(6)\qquad \begin{aligned} &(a) \quad r'' = \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3 - 4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots, \\ &(b) \quad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2 q - q^3}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots, \\ &(c) \quad v' = \frac{c}{r^2}. \end{aligned}$$

The first two of these equations are independent of the third so that r and q may be considered as rectangular coördinates in a revolving plane which passes always through the z-axis and through the particle itself. The problem is thus reduced to the consideration of the motion in this plane, for, when r is known, v is obtained by a simple quadrature from (6c).

§ 3. *Surfaces of zero velocity.*

The velocity integral in the revolving plane is

$$r'^2 + q'^2 = \frac{2}{(r^2+q^2)^{\frac{1}{2}}} + \frac{2}{3}\,\frac{r^2-2q^2}{(r^2+q^2)^{\frac{5}{2}}}\,\theta_1^2\mu^2 + \cdots - \frac{c^2}{r^2} + c_2.$$

If we put the velocity equal to zero the resulting equation represents a two-parameter family of curves. For assigned values of the parameters c and c_2 there is defined a certain curve in the revolving plane. On one side of this curve the motion is real while on the other side it is imaginary. For values of $c_2 < 0$ this curve is closed, and the motion is real on the inside. As the plane revolves this curve generates a surface of the general form of an anchor ring. For $\mu^2 = 0$ this curve belongs to the ordinary two-body problem, and its equation is

$$\frac{2}{(r^2+q^2)^{\frac{1}{2}}} - \frac{c^2}{r^2} + c_2 = 0\,.$$

The motion is elliptic, parabolic, or hyperbolic according as c_2 is negative, zero, or positive. Putting

$$r = \rho\cos\phi, \qquad q = \rho\sin\phi,$$

we find, on solving for ρ,

$$\rho = \frac{1}{c_2}\left[-1 \pm \sqrt{1+\frac{c_2c^2}{\cos^2\phi}}\,\right].$$

For negative values of c_2 this equation represents two closed ovals which do not enclose the origin. If $c_2c^2 = -1$ the ovals shrink upon the points $\rho = -1/c_2$, $\phi = 0$ and π. The corresponding orbit is therefore a circle in the equatorial plane. As c_2 approaches zero the ovals open out rapidly and approach the limiting curves

$$\rho = \frac{c^2}{2\cos^2\phi}.$$

For values of $c_2 > 0$ there is but one positive value for ρ, which is

$$\rho = \frac{1}{c_2}\left[-1 + \sqrt{1+\frac{c_2c^2}{\cos^2\phi}}\,\right].$$

If $c^2 \neq 0$ none of these curves crosses the axis $\phi = 90°$. But if $c = 0$ we have the circle $\rho = -2/c_2$ inside of which the motion is real when c_2 is negative.

For values of $\mu^2 \neq 0$, but sufficiently small, we can put

$$r = (\rho + \bar{\rho}) \cos \phi, \qquad q = (\rho + \bar{\rho}) \sin \phi,$$

and solve for $\bar{\rho}$ as a power series in μ^2. We find

$$\bar{\rho} = \frac{1}{3} \frac{2 - 3 \cos^2 \phi}{\rho(1 + c_2 \rho)} \theta_1^2 \mu^2 + \cdots,$$

which is the correction to be applied to the corresponding curves in the two-body problem.

Part I.

ORBITS WHICH REËNTER AFTER ONE REVOLUTION.

§ 4. *Symmetry.*

Returning to the differential equations (6*a*) and (6*b*) we observe that if we change

$$r \text{ into } + r, \qquad q \text{ into } - q, \qquad \tau \text{ into } - \tau,$$

the differential equations remain unchanged. Hence, if at some epoch, $\tau = \tau_0$,

$$r = \alpha, \qquad r' = 0,$$

$$q = 0, \qquad q' = \beta,$$

that is, if at the epoch $\tau = \tau_0$, the particle crosses the r-axis perpendicularly, it follows from the form of the differential equations that the orbit is symmetrical with respect to the r-axis and with respect to the epoch $\tau = \tau_0$. In other words r will be an even series in $(\tau - \tau_0)$ and q will be an odd series in $(\tau - \tau_0)$. If now at some other epoch, $\tau = \tau_0 + T$, the particle again crosses the r-axis perpendicularly the orbit will be symmetrical with respect to this epoch also. It is clear therefore that the orbit is a closed one, and that the motion in it is periodic, for, by hypothesis, at $\tau = \tau_0 - T$ it must have been at the same point and moving with the same velocity in the same direction. The motion is therefore periodic with the period $2T$. Hence with these initial values sufficient conditions for periodicity are that at $\tau = \tau_0 + T$

$$r' = q = 0.$$

From the integral of area, $v' = c/r^2$, it follows that if r is periodic v will have the form

$$v = \text{constant} \times \tau + \text{periodic terms}.$$

§ 5. *Existence of periodic orbits in the equatorial plane.*

If $q \equiv 0$ equations (6) reduce to

$$(8)\qquad \begin{aligned} &(a)\quad r'' = \frac{c^2}{r^3} - \frac{1}{r^2} - \frac{\theta_1^2\mu^2}{r^4} - \frac{\theta_2^2\mu^4}{r^6} - \cdots, \\ &(b)\quad v' = \frac{c}{r^2}. \end{aligned}$$

The first of these equations is independent of the second and can be integrated separately. It represents motion in a straight line in the revolving plane. It admits the constant solution

$$r_0 = 1, \qquad c_0^2 = 1 + \theta_1^2\mu^2 + \theta_2^2\mu^4 + \cdots,$$

which represents a point in the revolving plane, or a circle in the equatorial plane. In order to investigate the oscillations about this point let us put

$$r = 1 + e\rho, \qquad c^2 = c_0^2 + e\epsilon,$$

where ρ is a variable whose initial value is arbitrarily assigned, e is an arbitrary parameter corresponding to the eccentricity in the two-body problem, and ϵ is a parameter to be determined so that ρ shall be periodic.

On substituting these values (8*a*) and expanding as power series in e, the terms independent of e cancel out and it is possible to divide through by e. The equation then becomes

$$(9)\qquad \begin{aligned} \rho'' + [1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots]\,\rho = {} & \epsilon\,[1 - 3\rho e + 6\rho^2e^2 - 10\rho^3e^3 + \cdots] \\ & + \rho^2 e[3 - 4\theta_1^2\mu^2 - 15\theta_2^2\mu^4 + \cdots] \\ & + \rho^3e^2[-6 + 10\theta_1^2\mu^2 + 46\theta_2^2\mu^4 + \cdots] \\ & + \rho^4e^3[10 - 20\theta_1^2\mu^2 - 111\theta_2^2\mu^4 + \cdots] \\ & \cdots\cdots\cdots\cdots \end{aligned}$$

We can simplify this equation somewhat by dividing through by the coefficient of ρ in the left member and then substituting

$$\mathrm{T} = \tau\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}, \qquad \delta = \frac{\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots}.$$

The equation then becomes

$$(10)\qquad \begin{aligned} \frac{d^2\rho}{d\mathrm{T}^2} + \rho = {} & \delta\,[1 - 3\rho e + 6\rho^2e^2 - 10\rho^3e^3 + \cdots] + [3 + a_1]\,\rho^2 e \\ & + [-6 + a_2]\,\rho^3e^2 + [10 + a_3]\,\rho^4e^3 + \cdots, \end{aligned}$$

where

$$a_1 = -3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots,$$
$$a_2 = +4\theta_1^2\mu^2 + (4\theta_1^4 + 28\theta_2^2)\mu^4 + \cdots,$$
$$a_3 = -10\theta_1^2\mu^2 - (10\theta_1^4 + 81\theta_2^2)\mu^4 + \cdots.$$

Equation (10) can be integrated for ρ as a power series in δ and e. Let us take the initial values

$$\rho = -1, \qquad \rho' = 0.$$

By Poincaré's extension of Cauchy's theorem we know that the solution of equation (10) having the prescribed initial values exists and converges provided δ and e are sufficiently small, for all values of $\mathbf{T}$ in the interval $0 \leqq \mathbf{T} \leqq T$, where T is finite, but otherwise arbitrary. The condition for periodicity is simply

$$\text{(11)} \qquad \rho' = 0 \text{ at } \mathbf{T} = T.$$

If we choose $T = \pi$ an inspection of equation (10) shows that for $e = 0$ the solution for ρ is periodic with the period 2π whatever may be the value of δ, so that equation (11) must carry e as a factor. After integrating equation (10) we find that the condition (11) is

$$\text{(12)} \quad 0 = -[\tfrac{3}{2} + a_1]\pi\delta e - [\tfrac{3}{2} + \tfrac{5}{2}a_1 + \tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2]\pi e^2 + \text{higher degree terms.}$$

Aside from the factor e there remains an equation in which the linear terms in e and δ are present, and which may be solved. We find

$$\text{(13)} \qquad \delta = (-1 + \cdots)e + \cdots.$$

If this value of δ be substituted in equation (10) it will then admit periodic solutions for ρ as power series in e having the period 2π for all values of e sufficiently small. Furthermore the solution as a power series in e with the prescribed initial conditions is unique.

§ 6. *Existence of periodic orbits not lying in the equatorial plane.*

For $\mu^2 = 0$ the differential equations (6) admit the circular solution

$$\text{(14)} \qquad \begin{array}{lll} r = 1, & q = 0, & v = \tau, \\ r' = 0, & q' = 0, & v' = 1, \end{array} \qquad c^2 = 1.$$

In order to investigate the existence of orbits not lying in the equatorial plane but having the period 2π for $\mu^2 \neq 0$, let us put

$$\text{(15)} \qquad r = 1 + \rho, \qquad q = 0 + \sigma, \qquad c^2 = 1 + \epsilon,$$

and take the initial conditions

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta\mu.$$

The conditions for periodicity (§ 4) are then

$$\rho' = \sigma = 0 \text{ at } \tau = \pi.$$

We have three arbitrary constants at our disposal, α, β, and ϵ, and two conditions to be satisfied. Hence one constant remains undetermined. We will therefore let β remain arbitrary and determine α and ϵ so as to satisfy the two conditions. After making the substitutions (15) and expanding, equations (6) become

$$(16)\quad \begin{aligned} \rho'' + \rho &= \epsilon - 3\rho\epsilon + 3\rho^2 + \tfrac{3}{2}\sigma^2 - \theta_1^2\mu^2 + 6\rho^2\epsilon - 6\rho^3 - 6\rho\sigma^2 \\ &\quad + 4\rho\theta_1^2\mu^2 + \text{higher degree terms,} \\ \sigma'' + \sigma &= 3\rho\sigma - 6\rho^2\sigma + \tfrac{3}{2}\rho^3 - \sigma\theta_1^2\mu^2 + \text{higher degree terms.} \end{aligned}$$

In order to integrate these equations let us put

$$(17)\quad \begin{aligned} \rho &= \sum_{i,j,k=0}^{\infty} \rho_{ijk}\epsilon^i\alpha^j\mu^k, \\ \sigma &= \sum_{i,j,k=1}^{\infty} \sigma_{ijk}\epsilon^i\alpha^j\mu^k. \end{aligned} \qquad (i+j+k>0),$$

The ρ_{ijk} and σ_{ijk} can be found by successive integrations, the constants of integration being determined so as to satisfy the initial conditions. In the series thus obtained take $\tau = \pi$. The two conditions for periodicity give us the two equations:

$$(18)\quad \begin{aligned} &(a)\ \rho'_{\tau=\pi} = 0 = a_1\epsilon^2 + a_2\epsilon\alpha + a_3\epsilon^3 + a_4\epsilon^2\alpha + a_5\epsilon\alpha^2 + a_6\alpha^3 + a_7\epsilon\mu^2 + a_8\alpha\mu^2 + a_9\mu^4 + \cdots, \\ &(b)\ \sigma_{\tau=\pi} = 0 = \beta\mu\,[b_1\epsilon + b_2\epsilon^2 + b_3\alpha^2 + b_4\epsilon\alpha + b_5\mu^2 + \cdots], \end{aligned}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero. Equation (18a) involves only the even powers of μ while (18b) involves only the odd powers. After dividing (18b) by $\beta\mu$ we can solve it for ϵ as a power series in α and μ^2 of the form

$$(19)\quad \epsilon = c_1\alpha^2 + c_2\mu^2 + c_3\alpha^3 + c_4\alpha\mu^2 + c_5\mu^4 + \cdots.$$

On substituting (19) in (18a) we obtain a series of the form

$$(20)\ (a)\quad 0 = d_1\alpha\mu^2 + d_2\alpha^3 + d_3\mu^4 + d_4\alpha^2\mu^2 + d_5\alpha^4 + \cdots.$$

If in this equation we make the substitution

$$(b)\quad \alpha = \left(\gamma - \frac{d_3}{d_1}\right)\mu^2,$$

we obtain

$$(c)\quad 0 = \mu^4[f_1\gamma + f_2\mu^2 + f_3\gamma\mu^2 + \cdots],$$

which can be solved uniquely for γ as a power series in μ^2. This solution substituted in (20*b*) gives α as a power series in μ^2, and this value of α substituted in (19) gives ϵ as a power series in μ^2. We thus have a solution

$$\alpha = \mu^2 P_1(\mu^2), \qquad \epsilon = \mu^2 P_2(\mu^2), \qquad \beta \text{ arbitrary}.$$

Newton's parallelogram shows that equation (20*a*) has two additional solutions, but as they are imaginary we do not stop to develop them.

§ 7. *Existence of periodic orbits in a meridian plane.*

If in equations (6) we take the area constant $c^2 = 0$ the motion of the particle is in a meridian plane; that is, the revolving plane has ceased to revolve and the orbit in this plane is the true orbit. After changing to polar coördinates by the substitution

$$r = p\cos\phi, \qquad q = p\sin\phi, \qquad c^2 = 0,$$

the differential equations become

$$(21)\qquad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

For $\mu^2 = 0$ we have the periodic solution

$$p = 1, \qquad \phi = \tau,$$

that is, a circle. For $\mu^2 \neq 0$ let us introduce ρ and σ by

$$p = 1 + \rho, \qquad \phi = \tau + \sigma,$$

with the initial values

$$\rho = \alpha, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta,$$

where α and β are two new arbitraries. By Poincaré's theorem ρ, ρ', σ and σ' are expansible as power series in α, β and μ^2 with τ entering the coefficients. The conditions for periodicity are that at $\tau = \pi$

$$\rho' = \sigma = 0.$$

If we perform the integration and then set $\tau = \pi$, we obtain the two following conditional equations:

$$(22)\qquad \begin{aligned} &(a)\quad \sigma_{\tau=\pi} = 0 = a_1\alpha + a_2\beta + a_3\alpha^2 + a_4\alpha\beta + a_5\beta^2 + a_6\mu^2 + \cdots, \\ &(b)\quad \rho'_{\tau=\pi} = 0 = \qquad\qquad b_3\alpha^2 + b_4\alpha\beta + b_5\beta^2 + 0\cdot\mu^2 + b_7\mu^4 + \cdots, \end{aligned}$$

where the a_i and b_i are constants which have been found from the actual integrations to be distinct from zero.

The first of these two equations can be solved for α as a power series in β and μ^2. This expression for α substituted in (b) gives rise to an equation of the form

$$(c) \qquad 0 = c_1\beta\mu^2 + c_2\beta^3 + c_3\beta^2\mu^2 + c_4\mu^4 + \cdots.$$

This equation has the same form as ($20a$) and can be solved in the same way, giving a unique real value for β as a power series in μ^2 vanishing with μ^2. This expression for β substituted in the equation for α gives α as a power series in μ^2, vanishing with μ^2. Therefore real periodic orbits exist for $\mu^2 \neq 0$ which are analytic continuations of circular orbits for $\mu^2 = 0$.

We have thus proved the existence of the three following classes of periodic orbits which have the period 2π, the generating orbits being circles:

I. Orbits lying in the equatorial plane;

II. Orbits not lying in the equatorial plane;

III. Orbits in a meridian plane.

§ 8. *Construction of periodic solutions in the equatorial plane.*

Let us first consider orbits in the equatorial plane. We retake the differential equations (8) and by means of the transformations there given we pass at once to equation (10), which is explicitly

$$(23) \qquad \begin{aligned} \frac{d^2\rho}{dT^2}+\rho=\delta[1-3\rho e+6\rho^2e^2+\cdots]&+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\rho^2 e\\ &+[-6+4\theta_1^2\mu^2+(4\theta_1^2+28\theta_2^2)\mu^4+\cdots]\rho^3e^2\\ &+[10-10\theta_1^2\mu^2-(10\theta_1^4+81\theta_2^2)\mu^4+\cdots]\rho^4e^3+\cdots. \end{aligned}$$

It was shown in equation (13) that δ can be expanded uniquely as a power series in e in such a manner that the solution for ρ as a power series in e will be periodic with the period 2π. Since the series is periodic with the same period for all values of e sufficiently small, it follows that the coefficient of each power of e is itself periodic. Since the solution exists and is unique, it must be possible to determine the δ uniquely by the condition that the solution is periodic. In the existence proof it was shown that δ vanishes with e. Therefore ρ and δ have the form

$$(24) \qquad \rho = \rho_0 + \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \cdots, \qquad \delta = \delta_1 e + \delta_2 e^2 + \delta_3 e^3 + \cdots.$$

The ρ_j are to be determined by the integration of equation (23) and by the initial values

$$\rho = -1, \qquad \frac{d\rho}{dT} = 0 \text{ at } T = 0.$$

The δ_j are to be determined in such a manner that the ρ_j shall be periodic.

Substituting (24) in (23) and equating the coefficients, we find:

$$
(25)\quad
\begin{aligned}
&(a)\quad \frac{d^2\rho_0}{dT^2} + \rho_0 = 0,\\
&(b)\quad \frac{d^2\rho_1}{dT^2} + \rho_1 = \delta_1 + [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]\rho_0^2,\\
&(c)\quad \frac{d^2\rho_2}{dT^2} + \rho_2 = \delta_2 - 3\rho_0\delta_1 + [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]2\rho_0\rho_1\\
&\qquad\qquad + [-6 + 4\theta_1^2\mu^2 + (4\theta_1^4 + 28\theta_2^2)\mu^4 + \cdots]\rho_0^3,\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(d)\quad \frac{d^2\rho_n}{dT^2} + \rho_n = \delta_n - 3\rho_0\delta_{n-1} + [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]2\rho_0\rho_{n-1}\\
&\qquad\qquad + f_n(\rho_0, \cdots, \rho_{n-2}),\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

These equations can be integrated in succession. The solution of (a) which satisfies the initial conditions is

$$
(26)\qquad \rho_0 = -\cos T.
$$

Since the initial conditions are independent of e, every ρ_j except ρ_0 must vanish at $T = 0$. Substituting (26) in (25b) and integrating, we have

$$
(27)\quad \rho_1 = \delta_1(1 - \cos T) + [3 - 3\theta_1^2\mu^2 - (\theta_1^4 + 6\theta_2^2)\mu^4 + \cdots]\,[\tfrac{1}{2} - \tfrac{1}{3}\cos T - \tfrac{1}{6}\cos 2T].
$$

The constants of integration in equation (27) have been determined so as to satisfy the initial conditions, but the constant δ_1 is as yet undetermined.

Substituting these values of ρ_0 and ρ_1 in (25c), we find

$$
(28)\quad
\begin{aligned}
\frac{d^2\rho_2}{dT^2} + \rho_2 = {} & [\delta_2 + \delta_1(3 - 3\theta_1^2\mu^2 + \cdots) + 3 - 6\theta_1^2\mu^2 + \cdots]\\
& + [\delta_1(-3 + 6\theta_1^2\mu^2 + \{2\theta_1^4 + 12\theta_2^2\}\mu^4 + \cdots)\\
& \qquad + (-3 + 12\theta_1^2\mu^2 - \{\tfrac{11}{2}\theta_1^2 - 9\theta_2^2\}\mu^4 + \cdots]\cos T\\
& + [\delta_1(3 - 3\theta_1^2\mu^2 + \cdots) + 3 - 18\theta_1^2\mu^2 + \cdots]\cos 2T\\
& + [3 - 4\theta_1^2\mu^2 + \cdots]\cos 3T.
\end{aligned}
$$

In order that the solution of this equation may be periodic the coefficient of $\cos T$ must be zero. This is the condition which determines δ_1 and consequently

$$
\delta_1 = -1 + 2\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 - \theta_2^2)\mu^4 + \cdots.
$$

With this value of δ_1 equation (28) becomes

$$\frac{d^2\rho_2}{d\mathbf{T}^2} + \rho_2 = [\delta_2 + 3\theta_1^2\mu^2 + \cdots] + [-9\theta_1^2\mu^2 + \cdots] \cos 2\mathbf{T} + [3 - 4\theta_1^2\mu^2 + \cdots] \cos 3\mathbf{T}.$$

The solution of this equation which satisfies the initial conditions is

$$(29) \quad \begin{aligned} \rho_2 = \delta_2(1 - \cos \mathbf{T}) &+ [3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{13}{2}\theta_1^2\mu^2 + \cdots] \cos \mathbf{T} \\ &+ [3\theta_1^2\mu^2 + \cdots] \cos 2\mathbf{T} + [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots] \cos 3\mathbf{T}. \end{aligned}$$

The constant δ_2 is as yet undetermined. It is determined by the periodicity condition for ρ_3 in the same manner that δ_1 was determined by the periodicity condition of ρ_2. Without giving the details of the computation its value is found to be

$$\delta_2 = -6\theta_1^2\mu^2 + \cdots.$$

This method of integration can be carried as far as may be desired. In orderto show this let us suppose that $\rho_0, \cdots, \rho_{n-1}$ have been computed and all the constants determined with the exception of δ_{n-1}. From $(25d)$ we have

$$(30) \quad \frac{d^2\rho_n}{d\mathbf{T}^2} + \rho_n = \delta_n - 3\rho_0\delta_{n-1} + [3 - 3\theta_1^2\mu^2 + \cdots]2\rho_0\rho_{n-1} + f_n(\rho_0, \cdots, \rho_{n-2}).$$

Here $f_n(\rho_0, \cdots, \rho_{n-2})$ is a polynomial in the ρ_j and contains only known terms. ρ_{n-1} depends upon δ_{n-1} in the following way:

$$\rho_{n-1} = \delta_{n-1}(1 - \cos \mathbf{T}) + \text{known terms.}$$

Equation (30) can therefore be written

$$\begin{aligned} \frac{d^2\rho_n}{d\mathbf{T}^2} + \rho_n = \delta_n + [3 - 3\theta_1^2\mu^2 + \cdots]\,\delta_{n-1} &+ [-3 + 6\theta_1^2\mu^2 + \cdots]\delta_{n-1} \cos \mathbf{T} \\ &+ [3 - 3\theta_1^2\mu^2 + \cdots] \cos 2\mathbf{T} + \text{known terms.} \end{aligned}$$

In order that the solution of this equation shall be periodic the coefficient of $\cos \mathbf{T}$ must be zero. This condition determines δ_{n-1}. The equation can then be integrated and the constants of integration will be determined by the conditions that

$$\rho_n = \frac{d\rho_n}{d\mathbf{T}} = 0 \text{ at } \mathbf{T} = 0.$$

Everything is then determined with the exception of δ_n, and we have then

$$\rho_n = (1 - \cos \mathbf{T})\,\delta_n + \text{known terms.}$$

Substituting the values of δ_1 and δ_2 in the solution as far as it has been computed, we find

$$\rho_0 = -\cos \mathrm{T},$$

$$\rho_1 = [\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 \cdots] + [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots] \cos \mathrm{T}$$
$$+ [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 \cdots] \cos 2\mathrm{T},$$

$$\rho_2 = [-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots] \cos \mathrm{T} + [3\theta_1^2\mu^2 + \cdots] \cos 2\mathrm{T}$$
$$+ [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots] \cos 3\mathrm{T},$$

$$\delta_1 = -1 + 2\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 - \theta_2^2)\mu^4 + \cdots,$$

$$\delta_2 = -6\theta_1^2\mu^2 + \cdots.$$

From these expressions we have the following series for $r = 1 + e\rho$:

$$
\begin{aligned}
(a) \quad r = 1 - e \cos \mathrm{T} + \{ & [\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\theta_1^4 - 4\theta_2^2)\mu^4 + \cdots] \\
& + [-\theta_1^2\mu^2 + (-\tfrac{7}{6}\theta_1^4 + 3\theta_2^2)\mu^4 + \cdots] \cos \mathrm{T} \\
(31) \qquad & + [-\tfrac{1}{2} + \tfrac{1}{2}\theta_1^2\mu^2 + (\tfrac{1}{6}\theta_1^4 + \theta_2^2)\mu^4 + \cdots] \cos 2\mathrm{T} \} e^2 \\
& + \{ [-3\theta_1^2\mu^2 + \cdots] + [\tfrac{3}{8} - \tfrac{1}{2}\theta_1^2\mu^2 + \cdots] \cos \mathrm{T} \\
& + [3\theta_1^2\mu^2 + \cdots] \cos 2\mathrm{T} + [-\tfrac{3}{8} + \tfrac{1}{2}\theta_1^2\mu^2 + \cdots] \cos 3\mathrm{T} \} e^3 + \cdots.
\end{aligned}
$$

Substituting this value of r in the equation ($8b$)

$$r^2 \frac{dv}{d\mathrm{T}} = \sqrt{\frac{c_0^2 + e\epsilon}{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 \cdots}},$$

and integrating, we find

$$
\begin{aligned}
(b) \quad v - v_0 = \{ & [1 + \theta_1^2\mu^2 + (\tfrac{1}{2}\theta_1^4 + 2\theta_2^2)\mu^4 + \cdots] \\
& + [\theta_1^2\mu^2 + (-\tfrac{9}{4}\theta_1^4 + \tfrac{19}{2}\theta_2^2)\mu^4 + \cdots] e^2 + \cdots \} \mathrm{T} \\
(31) \qquad & + \{ [2 + 2\theta_1^2\mu^2 + (\theta_1^4 + \theta_2^2)\mu^4 + \cdots] \sin \mathrm{T} \} e \\
& + \{ [2\theta_1^2\mu^2 + (\tfrac{13}{3}\theta_1^4 - 6\theta_2^2)\mu^4 + \cdots] \sin \mathrm{T} \\
& + [\tfrac{5}{4} + \tfrac{3}{4}\theta_1^2\mu^2 + (-\tfrac{1}{24}\theta_1^4 + \tfrac{3}{2}\theta_2^2)\mu^4 + \cdots] \sin 2\mathrm{T} \} e^2 + \cdots.
\end{aligned}
$$

Equations ($31a$) and ($31b$) are the periodic solutions sought. If we return to the symbols defined in the original differential equations by means of equations (4), with the additional notation

$$n\sqrt{1 - \theta_1^2\mu^2 - 3\theta_2^2\mu^4 + \cdots} = \nu,$$

we have the following expressions for the polar coördinates at any time t:

$$
\begin{aligned}
R = a \{ 1 - e \cos \nu t + [\tfrac{1}{2} - \tfrac{1}{2}\cos 2\nu t] e^2 + [\tfrac{3}{8}\cos \nu t - \tfrac{3}{8}\cos 3\nu t] e^3 + \cdots \\
+ \frac{b^2}{a^2}\mu^2 [(\tfrac{3}{20} - \tfrac{3}{10}\cos \nu t + \tfrac{3}{20}\cos 2\nu t) e^2 + (-\tfrac{9}{10} - \tfrac{3}{20}\cos \nu t \\
+ \tfrac{9}{10}\cos 2\nu t + \tfrac{3}{20}\cos 3\nu t) e^3 + \cdots] + \frac{b^4}{a^4}\mu^4 [\cdots] + \cdots \},
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
v - v_0 = \nu t + 2\sin \nu t \cdot e + \tfrac{5}{4}\sin 2\nu t \cdot e^2 + \cdots + \frac{b^2}{a^2}\mu^2 [(\tfrac{3}{10} + \tfrac{3}{10}e^2 + \cdots)\nu t \\
+ (\tfrac{3}{5}\sin \nu t) e^2 + (\tfrac{3}{5}\sin \nu t + \tfrac{9}{40}\sin 2\nu t) e^2 + \cdots] + \frac{b^4}{a^4}\mu^4 [\cdots] + \cdots .
\end{aligned} \tag{33}
$$

These equations contain four arbitrary constants, a, e, v_0 and t_0. Since the differential equations of motion in the equatorial plane were of the fourth order these series, within the realm of their convergence, represent the general solution. The expression for the radius vector, R, is always periodic with the period $2\pi/\nu$. At the expiration of this period v has increased by the quantity

$$
2\pi \left[\frac{b^2}{a^2}\mu^2 (\tfrac{3}{10} + \tfrac{3}{10}e^2 + \cdots) + \cdots \right] = 2\pi\Theta
$$

in excess of 2π; that is, the line of apsides has rotated forwarded by this amount. If Θ is commensurable with unity the orbit is eventually closed geometrically. If $\Theta = I/J$, where I and J are integers relatively prime, then, at $t = 2J\pi/\nu$, $v = 2(I + J)\pi$, and the particle is at its initial position with its initial components of velocity. The particle has completed $I + J$ revolutions and the line of apsides has completed I revolutions. The mean sidereal period is

$$
P = \frac{2\pi}{\nu(1 + \Theta)}.
$$

This formula for the rotation of the line of apsides has an interesting application in the case of Jupiter's fifth satellite. On the hypothesis that Jupiter is homogeneous and taking its equatorial diameter as 90,190 miles and its polar diameter as 84,570 miles, the mean distance of the satellite as 112,500 miles, the eccentricity of its orbit as .006 and its mean sidereal period as $11^{\mathrm{h}}0^{\mathrm{m}}23^{\mathrm{s}}$, the above formula gives for the rotation of the line of apsides 1440° per year. The values derived from observations are somewhat discordant but are in the neighborhood of 880° per year. If we still keep the hypothesis that Jupiter is homogeneous in density and of the same oblateness as before, we are compelled, in order to relieve the discrepancy between theory and observation, to suppose that the value adopted for its polar radius was about 9,000 miles too great. From the large reduction required it is clear that the hypothesis of homogeneity is very much in error.

§ 9. *Construction of periodic solutions not lying in the equatorial plane.*

By means of the area integral the problem has been reduced to the two equations

$$
(35) \qquad
\begin{aligned}
r'' &= \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3 - 4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots, \\
q'' &= -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2 q - 2q^3}{(r^2+q^2)^{\frac{7}{2}}}\theta_1^2\mu^2 + \cdots.
\end{aligned}
$$

After the solution of these equations has been obtained the third coördinate is found from the equation

$$v' = \frac{c}{r^2}.$$

We have already proved [equations (14) to (20)] the existence of periodic solutions of these equations of the following type:

$$
(36) \qquad
\begin{aligned}
r &= 1 + \rho_2\mu^2 + \rho_4\mu^4 + \cdots, \\
q &= q_1\mu + q_3\mu^3 + q_5\mu^5 + \cdots, \\
c^2 &= 1 + c_2\mu^2 + c_4\mu^4 + \cdots,
\end{aligned}
$$

with the initial conditions

$$r'(0) = q(0) = 0, \qquad q'(0) = \beta\mu,$$

β being arbitrary. We can therefore integrate equations (35) so as to satisfy these initial conditions and determine the c_j in such a manner as to render the solution periodic.

Substituting (36) in (35) gives the equations

$$
(37) \qquad
\begin{aligned}
0 = [\rho_2'' + \rho_2 - \tfrac{3}{2}q_1^2 - c_2 + \theta_1^2]\mu^2 + [\rho_4'' + \rho_4 - 3\rho_2^2 - 3q_1q_3 + 6\rho_2 q_1^2 \\
+ \tfrac{15}{8}q_1^4 + (3c_2 - 4\theta_1^2)\rho_2 - \tfrac{15}{2}\theta_1^2 q_1^2 - c_4]\mu^4 + \cdots,
\end{aligned}
$$

$$
(38) \qquad
\begin{aligned}
0 = [q_1'' + q_1]\mu + [q_3'' + q_3 - 3\rho_2 q_1 - \tfrac{3}{2}q_1^3 + 3\theta_1^2 q_1]\mu^3 + [q_5'' + q_5 - 3\rho_2 q_3 - \tfrac{9}{2}q_1^2 q_3 \\
+ 6\rho_2^2 q_1 - 3\rho_4 q_1 + \tfrac{15}{2}\rho_2 q_1^3 + \tfrac{15}{8}q_1^5 + 3\theta_1^2 q_3 - 15\theta_1^2\rho_2 q_1 - \tfrac{25}{2}\theta_1^2 q_1^3]\mu^5 + \cdots.
\end{aligned}
$$

Equation (37) contains only even powers of μ while equation (38) contains only odd powers. For the integration we have:

Coefficient of μ.

$$(39) \qquad q_1'' + q_1 = 0.$$

The solution of this equation satisfying the initial conditions is

$$q_1 = \beta \sin \tau .$$

Coefficient of μ^2.

$$(40) \qquad \begin{aligned} \rho_2'' + \rho_2 &= \tfrac{3}{2} q_1^2 + c_2 - \theta_1^2, \\ &= (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) - \tfrac{3}{4}\beta^2 \cos 2\tau . \end{aligned}$$

The solution of this equation which satisfies the assigned initial conditions is

$$\rho_2 = (\tfrac{3}{4}\beta^2 + c_2 - \theta_1^2) + \alpha_2 \cos \tau + \tfrac{1}{4}\beta^2 \cos 2\tau .$$

The constant c_2 will be determined by the periodicity condition on q_3 to be

$$c_2 = 2\theta_1^2 - \beta^2,$$

and α_2 will be determined by the periodicity condition on ρ_4 to be

$$\alpha_2 = 0 .$$

If we anticipate these determinations we have

$$\rho_2 = (\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2 \cos 2\tau .$$

Coefficient of μ^3.

$$(41) \qquad \begin{aligned} q_3'' + q_3 &= q_1(3\rho_2 + \tfrac{3}{2} q_1^2 - 3\theta_1^2), \\ &= (3\beta^2 + 3c_2 - 6\theta_1^2)\beta \sin \tau + \tfrac{3}{2}\alpha_2\beta \sin 2\tau . \end{aligned}$$

In order that the solution shall be periodic it is necessary that the coefficient of $\sin \tau$ shall be zero. Therefore

$$c_2 = 2\theta_1^2 - \beta^2 .$$

Substituting this value and integrating, we find

$$q_3 = \beta_3 \sin \tau - \tfrac{1}{4}\alpha_2 \beta \sin 2\tau .$$

From the initial conditions we must have $q_3'(0) = 0$ and therefore

$$\beta_3 = \tfrac{1}{2}\alpha_2 \beta .$$

But it will be shown in the next step that $\alpha_2 = 0$, and consequently

$$q_3 \equiv 0 .$$

Coefficient of μ^4.

$$(42) \quad \rho_4'' + \rho_4 = 3\rho_2^2 + 3q_1q_3 - 6\rho_2 q_1^2 - \tfrac{15}{8} q^4 + (4\theta_1^2 - 3c_2)\rho_2 + \tfrac{15}{2}\theta_1^2 q_1^2 + c_4 .$$

Before expanding the right member of this equation we will avoid useless labor by first examining the coefficient of $\cos \tau$ which we know must be zero from the periodicity condition. We notice first that terms in the coefficient of $\cos \tau$ can arise only through terms involving ρ_2 and q_3, and second that all such terms carry α_2 as a factor. Since no other arbitrary enters the coefficient, we must take $\alpha_2 = 0$. It can be shown by induction that the constant of integration α_{2i} (the coefficient of $\cos \tau$) which arises in the integration of ρ_{2i} is determined by the periodicity condition on $\rho_{2(i+1)}$, and further that its value is always zero. The proof is omitted for the sake of brevity.

Substituting the value $\alpha_2 = 0$ in ρ_2 and q_3 and expanding the right member of (42), we find

$$\rho_4'' + \rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{3}{16}\beta^4]\cos 2\tau + \tfrac{15}{64}\beta^4 \cos 4\tau.$$

The solution of this equation satisfying the initial conditions and the properties just mentioned is

$$\rho_4 = [c_4 + \theta_1^2 + \tfrac{11}{4}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4]\cos 2\tau - \tfrac{1}{64}\beta^4 \cos 4\tau.$$

Anticipating the value of c_4 found below, we have

$$\rho_4 = [-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4] + [-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^2]\cos 2\tau - \tfrac{1}{64}\beta^4 \cos 4\tau.$$

Coefficient of μ^5.

$$\begin{aligned} q_5'' + q_5 &= 5\rho_2 q_3 + \tfrac{9}{2}q_1^2 q_3 - 6\rho_2^2 q_1 + 3\rho_4 q_1 - \tfrac{15}{2}\rho_2 q_1^2 - \tfrac{15}{8}q_1^5 \\ (43)\qquad &\qquad - 3\theta_1^2 q_3 + 15\theta_1^2\rho_2 q_1 + \tfrac{25}{2}\theta_1^2 q_1^3, \\ &= [3c_4 + 12\theta_1^4 + 11\theta_1^2\beta^2]\beta \sin \tau - \theta_1^2\beta^3 \sin 3\tau. \end{aligned}$$

From the periodicity condition we have

$$c_4 = -4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2.$$

Integrating and imposing the initial conditions, we find

$$q_5 = -\tfrac{3}{8}\theta_1^2\beta^3 \sin \tau + \tfrac{1}{8}\theta_1^2\beta^3 \sin 3\tau.$$

This is sufficient to make evident the general character of the series. The r-equation contains only even multiples of τ, and the q-equation contains only odd multiples. The r-equation contains only even powers of μ and of τ. The q-equation is odd in both these respects. The series are therefore triply even and odd.

Collecting the various coefficients we have the following series:

$$(44)\ \begin{cases} (a) & r = 1 + [(\theta_1^2 - \tfrac{1}{4}\beta^2) + \tfrac{1}{4}\beta^2 \cos 2\tau]\mu^2 + [(-3\theta_1^4 - \tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{3}{64}\beta^4) \\ & \qquad + (-\tfrac{1}{12}\theta_1^2\beta^2 + \tfrac{1}{16}\beta^4)\cos 2\tau - \tfrac{1}{64}\beta^4 \cos 4\tau]\mu^4 + \cdots, \\ (b) & q = [\beta \sin \tau]\mu + [0]\mu^3 + [-\tfrac{3}{8}\theta_1^2\beta^3 \sin \tau + \tfrac{1}{8}\theta_1^2\beta^3 \sin 3\tau]\mu^5 + \cdots, \\ (c) & v - v_0 = [1 - \theta_1^2\mu^2 - (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots]\tau + [-\tfrac{1}{4}\beta^2 \sin 2\tau]\mu^2 \\ & \qquad + [(\tfrac{11}{12}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{64}\beta^4 \sin 4\tau]\mu^4 + \cdots, \\ (d) & c^2 = 1 + [2\theta_1^2 - \beta^2]\mu^2 + [-4\theta_1^4 - \tfrac{11}{3}\theta_1^2\beta^2]\mu^4 + \cdots. \end{cases}$$

This solution contains four arbitrary constants, a, β, v_0 and τ_0. As is shown by equation (44c) the nodes regress, the rate of regression being

$$[\theta_1^2\mu^2 + (\tfrac{3}{2}\theta_1^4 + \tfrac{1}{6}\theta_1^2\beta^2)\mu^4 + \cdots].$$

The generating orbit of these solutions is a circle in the equatorial plane. A circle having any assigned inclination might have been used, e. g.,

$$(45)\quad r = \sqrt{1 - s^2\sin^2\tau}, \qquad q = s\sin\tau, \qquad v = \tan^{-1}[\sqrt{1 - s^2}\tan\tau],$$

where s is the sine of the inclination. The solution thus obtained would have been identical with (44). If we expand (45) as power series in s and then put $s = \beta\mu$, it will be found that the terms thus obtained are identical with the terms independent of θ_1^2 in the solution which has been worked out. It might therefore be of assistance in the physical interpretation of the constants to put $\beta\mu = s$ in the series (44).

§ 10. *Construction of periodic solutions in a meridian plane.*

When the constant c is zero the motion is in a meridian plane. The equations of motion are (21)

$$(46)\quad \begin{aligned} p'' - p\phi'^2 + \frac{1}{p^2} &= -\frac{-\frac{3}{4} + \frac{3}{2}\cos 2\phi + \frac{1}{4}\cos 4\phi}{p^4}\theta_1^2\mu^2 + \cdots, \\ p\phi'' + 2p'\phi' &= -\frac{\frac{1}{2}\sin 2\phi - \frac{1}{4}\sin 4\phi}{p^4}\theta_1^2\mu^2 + \cdots. \end{aligned}$$

We have already shown the existence of periodic solutions of these equations as power series in μ^2, which for $\mu = 0$ reduce to the circle $p = 1$, $\phi = \tau$. Let us put then

$$p = 1 + p_2\mu^2 + p_4\mu^4 + \cdots, \qquad \phi = \tau + \phi_2\mu^2 + \phi_4\mu^4 + \cdots.$$

Substituting these expressions in (46), expanding and collecting the coefficients of the various powers of μ, we find:

$$(47)\quad \begin{aligned} 0 = {} & [p_2'' - 3p_2 - 2\phi_2' - \tfrac{3}{4}\theta_1^2 + \tfrac{3}{2}\theta_1^2\cos 2\tau + \tfrac{1}{4}\theta_1^2\cos 4\tau]\mu^2 + [p_4'' - 3p_4 - 2\phi_4' - 2p_2\phi_2' \\ & - \phi_2'^2 + 3p_2^2 + (3 - 6\cos 2\tau - \cos 4\tau)\theta_1^2 p_2 + (-3\sin 2\tau - \sin 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots, \\ 0 = {} & [\phi_2'' + 2p_2' + \tfrac{1}{2}\theta_1^2\sin 2\tau - \tfrac{1}{4}\theta_1^2\sin 4\tau]\mu^2 + [\phi_4'' + 2p_4' + \phi_2''p_2 + 2p_2'\phi_2' \\ & + (-2\sin 2\tau + \sin 4\tau)\theta_1^2 p_2 + (\cos 2\tau - \cos 4\tau)\theta_1^2\phi_2]\mu^4 + \cdots. \end{aligned}$$

The initial conditions are

$$p'(0) = \phi(0) = 0.$$

Proceeding to the integration we have:

Coefficient of μ^2.

$$(48)\qquad \begin{aligned} &(a)\quad p_2'' - 3p_2 - 2\phi_2' = \tfrac{3}{4}\theta_1^2 - \tfrac{3}{2}\theta_1^2\cos 2\tau - \tfrac{1}{4}\theta_1^2\cos 4\tau,\\ &(b)\qquad\qquad \phi_2'' + 2p_2' = -\tfrac{1}{2}\theta_1^2\sin 2\tau + \tfrac{1}{4}\theta_1^2\sin 4\tau. \end{aligned}$$

By integrating (b) once we have

$$(c)\qquad \phi_2' = -2p_2 + \tfrac{1}{4}\theta_1^2\cos 2\tau - \tfrac{1}{16}\theta_1^2\cos 4\tau + c_1.$$

Substituting this value of ϕ_2' in (a), we find

$$(d)\qquad p_2'' + p_2 = (2c_1 + \tfrac{5}{4}\theta_1^2) - \theta_1^2\cos 2\tau - \tfrac{3}{8}\theta_1^2\cos 4\tau.$$

The integration of this equation gives

$$(e)\qquad p_2 = (2c_1 + \tfrac{5}{4}\theta_1^2) + c_2\sin\tau + c_3\cos\tau + \tfrac{1}{3}\theta_1^2\cos 2\tau + \tfrac{1}{40}\theta_1^2\cos 4\tau.$$

Since $p_2' = 0$ at $\tau = 0$ we must take $c_2 = 0$. Then substituting this value of p_2 in (c) and integrating, we get

$$(f)\qquad \phi_2 = (-3c_1 - \tfrac{5}{2}\theta_1^2)\tau - 2c_3\sin\tau - \tfrac{5}{24}\theta_1^2\sin 2\tau - \tfrac{9}{320}\theta_1^2\sin 4\tau + c_4.$$

From the initial conditions ϕ_2 must be zero when $\tau = 0$. Therefore $c_4 = 0$. Since it must also be periodic, $c_1 = -\tfrac{5}{6}\theta_1^2$. All of the constants of integration are now determined except c_3 which will be determined by the periodicity condition on p_4.

The differential equations for p_4 and ϕ_4 are the same as for p_2 and ϕ_2 except in the right members. The process of integration is therefore the same. In the right members only even multiples of τ occur except terms carrying the undetermined c_3 as a factor. In the equation corresponding to (48d) there will be a term in $\cos\tau$ carrying c_3 as a factor. But the integration of this term will be non-periodic unless $c_3 = 0$. Put then $c_3 = 0$; the integration proceeds just as before and the constants are determined in the same manner. This argument will be repeated in the coefficients of μ^6 and so on for all higher powers.* Therefore no odd multiples of τ can occur in the solution. We have, therefore,

$$(49)\qquad \begin{aligned} p &= 1 + [-\tfrac{1}{4} + \tfrac{1}{3}\cos 2\tau + \tfrac{1}{40}\cos 4\tau]\theta_1^2\mu^2 + \cdots,\\ q &= \tau + [-\tfrac{5}{24}\sin 2\tau - \tfrac{9}{320}\sin 4\tau]\theta_1^2\mu^2 + \cdots. \end{aligned}$$

* The differential equations arising at the successive steps are of the same type as the first two equations of (32), p. 555, in Professor Moulton's paper on periodic solutions of the problem of three bodies, these Transactions, vol. 7 (1906). Consequently his general formulas for the coefficients of the solutions, equations (42), loc. cit., apply here.

Since the series involve only even multiples of τ the orbits are symmetrical with respect to both the r-axis and the q-axis.

This completes the formal construction of the solutions of which the existence was proved in §§ 5, 6 and 7.

PART II.

ORBITS REËNTRANT ONLY AFTER MANY REVOLUTIONS.

§ 11. *The differential equations.*

The orbits which we have previously considered have had the common property of involving only the period 2π. Since this period is independent of the oblateness of the spheroid the derivation of these orbits has been relatively simple. We shall proceed now to investigate a class of orbits which involves beside the period 2π another period $2\pi/\lambda$, where λ is a function of the oblateness of the spheroid, the inclination of the orbit and the mean distance of the particle. We shall start from the solution which involves an arbitrary inclination. Into this solution four arbitrary constants were introduced, viz., inclination, mean distance, longitude of node and the epoch. Two more arbitraries are necessary for a complete solution, viz., constants corresponding to the eccentricity and to the longitude of perihelion from the node. In what follows we shall introduce the constant corresponding to the eccentricity, but we shall still project the particle from an apse at the node.

We have found for the differential equations a certain solution (44) which we may write

$$r = \phi(\beta, \mu; \tau), \qquad q = \psi(\beta, \mu; \tau), \qquad c^2 = c_0^2,$$

which is symmetric with respect to the equatorial plane. That is to say, at $\tau = 0$ the particle is in the equatorial plane and its motion is perpendicular to the radius vector. Its initial distance is $\phi(0)$. Suppose now we change the initial distance slightly and also the initial velocity so that at $\tau = 0$

$$r(0) = \phi(0) + \alpha, \qquad q(0) = 0,$$

$$r'(0) = 0, \qquad q'(0) = \psi'(0) + \gamma,$$

and give an increment to the constant of areas so that $c^2 = c_0^2 + \epsilon$. Can we determine these three constants, α, γ and ϵ, in such a manner that the series for r and q shall be periodic? These series can be expressed by

$$r = \phi(\beta, \mu; \tau) + \rho, \qquad q = \psi(\beta, \mu; \tau) + \sigma, \qquad c^2 = c_0^2 + \epsilon.$$

If now we substitute these expressions in the differential equations (6) all the terms independent of ρ, σ and ϵ will drop out, and there will remain the following differential equations for ρ and σ:

$$
\begin{aligned}
&(a)\ \rho'' + \{1 + [(-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau]\,\mu^2 + [(20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4)\\
&\qquad + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau]\,\mu^4 + \cdots\}\,\rho + \{-3\beta\mu \sin\tau\\
&\qquad + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau]\,\mu^3 + \cdots\}\,\sigma\\
&\qquad + \epsilon\Big\{\{1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2\cos 2\tau]\,\mu^2 + [(15\theta_1^4 - \tfrac{1}{4}\theta_1^2\beta^2 + \tfrac{45}{64}\beta^4)\\
&\qquad + (\tfrac{13}{4}\theta_1^2\beta^2 - \tfrac{15}{16}\beta^4)\cos 2\tau + \tfrac{15}{4}\beta^4\cos 4\tau]\,\mu^4 + \cdots\}\\
&\qquad + \{-3 + [(12\theta_1^2 - 3\beta^2) + 3\beta^2\cos 2\tau]\,\mu^2 + \cdots\}\,\rho + \cdots\Big\}\\
&\quad = \{3 + [(-16\theta_1^2 + 6\beta^2) - 12\beta^2\cos 2\tau]\mu^2 + [(90\theta_1^4 + \tfrac{59}{4}\theta_1^2\beta^2 + \tfrac{69}{64}\beta^4)\\
(50)\quad&\qquad + (\tfrac{1}{4}\theta_1^2\beta^2 - \tfrac{63}{16}\beta^4)\cos 2\tau + \tfrac{183}{64}\beta^4\cos 4\tau]\,\mu^4 + \cdots\}\,\rho^2 + \{-12\beta\mu\sin\tau\\
&\qquad + [(-90\theta_1^2\beta + \tfrac{45}{4}\beta^3)\sin\tau - \tfrac{15}{4}\beta^3\sin 3\tau]\,\mu^3 + \cdots\}\,\rho\sigma\\
&\qquad + \{\tfrac{3}{2} + [(\tfrac{3}{2}\theta_1^2 - \tfrac{33}{8}\beta^2) + \tfrac{33}{8}\beta^2\cos 2\tau]\,\mu^2 + \cdots\}\,\sigma^2 +\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(b)\ \sigma'' + \{1 + [-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2\cos 2\tau]\mu^2 + [-4\theta_1^2\beta^2 + 7\theta_1^2\beta^2\cos 2\tau]\,\mu^4 + \cdots\}\,\sigma\\
&\qquad + \{-3\beta\mu\sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3\sin 3\tau]\mu^3 + \cdots\}\,\rho\\
&\quad = \{-6\beta\mu\sin\tau + [(-21\theta_1^2\beta + \tfrac{63}{8}\beta^3)\sin\tau - \tfrac{21}{8}\beta^3\sin 3\tau]\mu^3 + \cdots\}\rho^2\\
&\qquad + \{3 + [(3\theta_1^2 - \tfrac{33}{4}\beta^2) + \tfrac{33}{4}\beta^2\cos 2\tau]\mu^2 + \cdots\}\rho\sigma + \{\tfrac{9}{2}\beta\mu\sin\tau + \cdots\}\sigma^2 +\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

In the first of these equations the coefficients of all terms containing odd powers of σ involve only odd powers of μ and sines of odd multiples of τ. All other coefficients involve only even powers of μ and cosines of even multiples of τ. In the second equation the coefficient of every odd power of σ involves only even powers of μ and cosines of even multiples of τ. All other coefficients involve only odd powers of μ and sines of odd multiples of τ. These properties play an important rôle throughout the entire discussion.

§ 12. *The equations of variation.*

Considering merely the linear terms of the differential equations (50) for ρ and σ we have the equations of variation:

$$
\begin{aligned}
&(a)\ \rho'' + \{1 + [(-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2\cos 2\tau]\,\mu^2\\
&\qquad + [(20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4\cos 4\tau]\,\mu^4 + \cdots\}\,\rho\\
(51)\quad&\qquad + \{-3\beta\mu\sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3\sin 3\tau]\,\mu^3 + \cdots\}\,\sigma = 0.\\
&(b)\ \sigma'' + \{1 + [-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2\cos 2\tau]\,\mu^2 + \cdots\}\,\sigma\\
&\qquad + \{-3\beta\mu\sin\tau + [(-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3\sin 3\tau]\,\mu^3 + \cdots\}\,\rho = 0.
\end{aligned}
$$

These equations admit a solution of the form *

$$\rho_j = A_j e^{\lambda_j \tau} \phi_j(\tau),$$
$$\sigma_j = A_j e^{\lambda_j \tau} \psi_j(\tau) \qquad (j=1, \cdots, 4)$$

where λ_j is a root of the fundamental equation and where $\phi_j(\tau)$ and $\psi_j(\tau)$ are periodic functions of τ with the period 2π. The four values of the λ_j (real or imaginary) are associated in pairs, equal in value but of opposite sign.† From the fact that τ does not occur explicitly in the original equations (1) it is known *à priori* that one pair of the λ_j has the value 0,‡ and consequently, if we choose the notation so that $\lambda_3 = \lambda_4 = 0$, the two corresponding solutions will have the forms

$$\rho_3 = A_3 \phi_3(\tau), \qquad \rho_4 = A_4[\phi_4(\tau) + \tau\phi_3(\tau)],$$
$$\sigma_3 = A_3 \psi_3(\tau), \qquad \sigma_4 = A_4[\psi_4(\tau) + \tau\psi_3(\tau)].$$

We shall consider first the two solutions in which the λ_j are not zero. Let us first substitute in (51) the forms

$$\rho = e^{i\lambda\tau}\phi(\tau) \qquad (i = \sqrt{-1}),$$
$$\sigma = e^{i\lambda\tau}\psi(\tau).$$

After dividing out the exponential there remains

$$(52) \quad \begin{aligned} &\phi'' + 2i\lambda\phi' + [1 - \lambda^2 + a_2\mu^2 + a_4\mu^4 + \cdots]\phi + [a_1\mu + a_3\mu^3 + \cdots]\psi = 0, \\ &\psi'' + 2i\lambda\psi' + [1 - \lambda^2 + b_2\mu^2 + b_4\mu^4 + \cdots]\psi + [a_1\mu + a_3\mu^3 + \cdots]\phi = 0, \end{aligned}$$

where

$$a_1 = -3\beta \sin\tau,$$

$$a_3 = (-3\theta_1^2\beta + \tfrac{9}{8}\beta^3)\sin\tau - \tfrac{3}{8}\beta^3 \sin 3\tau,$$

$$a_2 = (-4\theta_1^2 + \tfrac{3}{2}\beta^2) - \tfrac{9}{2}\beta^2 \cos 2\tau,$$

$$a_4 = (20\theta_1^4 + 6\theta_1^2\beta^2 + \tfrac{3}{8}\beta^4) + (\theta_1^2\beta^2 - \tfrac{3}{2}\beta^4)\cos 2\tau + \tfrac{9}{8}\beta^4 \cos 4\tau,$$

$$b_2 = -\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\beta^2 \cos 2\tau,$$

$$b_4 = \text{a sum of cosines of even multiples of } \tau.$$

With respect to equations (52) it is known that ϕ and ψ are periodic with the

* Floquet, Annales Scientifiques de l'École Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

† See *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1, p. 193.

‡ Ibid., p. 187.

period 2π, and that λ vanishes with μ since the problem then reduces to the two-body problem in which the characteristic exponents are all zero. It can be shown that ϕ, ψ and λ are expansible as power series in μ of the form

$$\phi = \sum_{j=0}^{\infty} \phi_j \mu^j, \qquad \psi = \sum_{j=0}^{\infty} \psi_j \mu^j, \qquad \lambda = \sum_{j=1}^{\infty} \lambda_j \mu^j.$$

Substituting these expressions in (52) we find:

Coefficient of μ^0.

$$\phi_0'' + \phi_0 = 0,$$

$$\psi_0'' + \psi_0 = 0.$$

Therefore

$$(53) \qquad \begin{aligned} \phi_0 &= \alpha_1^{(0)} \cos \tau + \alpha_2^{(0)} \sin \tau, \\ \psi_0 &= \gamma_1^{(0)} \cos \tau + \gamma_2^{(0)} \sin \tau. \end{aligned}$$

Coefficient of μ.

$$(54) \qquad \begin{aligned} \phi_1'' + \phi_1 &= -2i\lambda_1 \phi_0' - a_1 \psi_0, \\ \psi_1'' + \psi_1 &= -2i\lambda_1 \psi_0' - a_1 \phi_0. \end{aligned}$$

Since the periodicity conditions demand that the coefficients of $\cos \tau$ and $\sin \tau$ in the right members be zero, we must take $\lambda_1 = 0$. We get then

$$(55) \qquad \begin{aligned} \phi_1'' + \phi_1 &= \tfrac{3}{2}\beta\gamma_2^{(0)} - \tfrac{3}{2}\beta\gamma_2^{(0)} \cos 2\tau + \tfrac{3}{2}\beta\gamma_1^{(0)} \sin 2\tau, \\ \psi_1'' + \psi_1 &= \tfrac{3}{2}\beta\alpha_2^{(0)} - \tfrac{3}{2}\beta\alpha_2^{(0)} \cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(0)} \sin 2\tau. \end{aligned}$$

Integrating we have

$$(56) \qquad \begin{aligned} \phi_1 &= \alpha_1^{(1)} \cos \tau + \alpha_2^{(1)} \sin \tau + \tfrac{3}{2}\beta\gamma_2^{(0)} + \tfrac{1}{2}\beta\gamma_2^{(0)} \cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(0)} \sin 2\tau, \\ \psi_1 &= \gamma_1^{(1)} \cos \tau + \gamma_2^{(1)} \sin \tau + \tfrac{3}{2}\beta\alpha_2^{(0)} + \tfrac{1}{2}\beta\alpha_2^{(0)} \cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(0)} \sin 2\tau. \end{aligned}$$

Coefficient of μ^2.

$$(57) \qquad \begin{aligned} \phi_2'' + \phi_2 &= -2i\lambda_2 \phi_0' - a_2 \phi_0 - a_1 \psi_1, \\ \psi_2'' + \psi_2 &= -2i\lambda_2 \psi_0' - b_2 \psi_0 - a_1 \phi_1; \end{aligned}$$

or, expanded,

$$(58) \qquad \begin{aligned} \phi_2'' + \phi_2 = {} & \tfrac{3}{2}\beta\gamma_2^{(1)} - \tfrac{3}{2}\beta\gamma_2^{(1)} \cos 2\tau + \tfrac{3}{2}\gamma_1^{(1)} \sin 2\tau + (4\theta_1^2 \alpha_2^{(0)} + 2i\lambda_2 \alpha_1^{(0)}) \sin \tau \\ & + (4\theta_1^2 \alpha_1^{(0)} - 2i\lambda_2 \alpha_2^{(0)}) \cos \tau + 3\beta^2 \alpha_2^{(0)} \sin 3\tau + 3\beta^2 \alpha_1^{(0)} \cos 3\tau, \\ \psi_2 + \psi_2 = {} & \tfrac{3}{2}\beta\alpha_2^{(1)} + (\tfrac{15}{2}\beta^2 \gamma_2^{(0)} + 2i\lambda_2 \gamma_1^{(0)}) \sin \tau + 2i\lambda_2 \gamma_2^{(0)} \cos \tau \\ & + \tfrac{3}{2}\beta\alpha_1^{(1)} \sin 2\tau - \tfrac{3}{2}\beta\alpha_2^{(1)} \cos 2\tau. \end{aligned}$$

In order to satisfy the periodicity condition we must have

$$
(59) \qquad \begin{aligned} 2i\lambda_2\alpha_1^{(0)} + 4\theta_1^2\alpha_2^{(0)} &= 0, & 2i\lambda_2\gamma_1^{(0)} + \tfrac{15}{2}\beta^2\gamma_2^{(0)} &= 0, \\ 4\theta_1^2\alpha_1^{(0)} - 2i\lambda_2\alpha_2^{(0)} &= 0, & + 2i\lambda_2\gamma_2^{(0)} &= 0. \end{aligned}
$$

The equations of the second column are satisfied by taking

$$\gamma_1^{(0)} = \gamma_2^{(0)} = 0.$$

Solving the other two we find

$$\lambda_2 = \pm 2\theta_1^2, \qquad \alpha_1^{(0)} - i\alpha_2^{(0)} = 0.$$

Equations (59) can also be satisfied by taking

$$\lambda_2 = \alpha_1^{(0)} = \alpha_2^{(0)} = \gamma_2^{(0)} = 0, \qquad \gamma_1^{(0)} = \text{arbitrary},$$

but this would lead to the development of the solutions in which the characteristic exponent is zero, and these solutions will be discussed later.

It was known at the outset that there were two values of λ equal numerically but of opposite sign. We will choose the one with the positive sign. The solution for the negative λ can be derived from the solution for the positive λ. The condition $\alpha_1^{(0)} - i\alpha_2^{(0)} = 0$ still leaves us with an arbitrary constant. Since the equations are linear this constant will enter the solution linearly and may therefore be taken equal to unity. The arbitrary constant is restored in (79) after the solutions are completely developed. We will take then $\alpha_1^{(0)} = 1$ which makes $\alpha_2^{(0)} = -i$. Consequently

$$
(60) \qquad \phi_0 = \cos\tau - i\sin\tau, \qquad \psi_0 = 0.
$$

Integrating (58) with these values, we get

$$
(61) \qquad \begin{aligned} \phi_2 &= \tfrac{3}{2}\beta\gamma_2^{(1)} + \alpha_1^{(2)}\cos\tau + \alpha_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(1)}\sin 2\tau \\ &\qquad + 3\beta^2\cos 3\tau - 3\beta^2 i\sin 3\tau, \\ \psi_2 &= \tfrac{3}{2}\beta\alpha_2^{(1)} + \gamma_1^{(2)}\cos\tau + \gamma_2^{(2)}\sin\tau + \tfrac{1}{2}\beta\alpha_2^{(1)}\cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(1)}\sin 2\tau. \end{aligned}
$$

Coefficient of μ^3.

$$
(62) \qquad \begin{aligned} \phi_3'' + \phi_3 &= \tfrac{3}{2}\beta\gamma_2^{(2)} + [-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)})]\cos\tau \\ &\quad + [2i\lambda_3 + 4\theta_1^2(\alpha_2^{(1)} + i\alpha_1^{(1)})]\sin\tau + \tfrac{3}{2}\beta\gamma_1^{(2)}\sin 2\tau - \tfrac{3}{2}\beta\gamma_2^{(2)}\cos 2\tau, \\ \psi_3'' + \psi_3 &= [\tfrac{3}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + [-2i\lambda_2\gamma_2^{(1)} + \tfrac{3}{4}\beta^2\gamma_1^{(1)}]\cos\tau \\ &\quad + [2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)}]\sin\tau + [-\tfrac{3}{2}\alpha_2^{(2)}\beta + \tfrac{11}{2}i\theta_1^2\beta + \tfrac{21}{16}i\beta^3]\cos 2\tau \\ &\quad + [\tfrac{3}{2}\alpha_1^{(2)}\beta + \tfrac{11}{2}\theta_1^2\beta - \tfrac{9}{16}\beta^3]\sin 2\tau - \tfrac{3}{4}\beta^2\gamma_1^{(1)}\cos 3\tau - \tfrac{3}{4}\beta^2\gamma_2^{(1)}\sin 3\tau. \end{aligned}
$$

From the periodicity conditions we must have

$$-2\lambda_3 + 4\theta_1^2(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0, \qquad \tfrac{3}{4}\beta^2\gamma_1^{(1)} - 2i\lambda_2\gamma_2^{(1)} = 0,$$
$$2i\lambda_3 + 4\theta_1^2(i\alpha_1^{(1)} + \alpha_2^{(1)}) = 0, \qquad 2i\lambda_2\gamma_1^{(1)} + \tfrac{9}{4}\beta^2\gamma_2^{(1)} = 0.$$

The last two equations can be satisfied only if $\gamma_1^{(1)} = \gamma_2^{(1)} = 0$. The first two can be satisfied only if $\lambda_3 = (\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$. The condition $(\alpha_1^{(1)} - i\alpha_2^{(1)}) = 0$ again gives us an arbitrary constant. Then by (56) $\phi_1 = c(\cos\tau - i\sin\tau)$, but this is the same as ϕ_0 multiplied by $c\mu$. That is, the solution is repeating itself one degree higher in μ and this, of course, should be expected since the equations are linear, so that any solution multiplied by any power of μ must satisfy them. We are at liberty then to choose the arbitrary $c = 0$, which is the same as choosing $\alpha_1^{(1)} = \alpha_2^{(1)} = 0$. Integrating (62) with these values, we find

$$\phi_3 = \tfrac{3}{2}\beta\gamma_2^{(2)} + \alpha_1^{(3)}\cos\tau + \alpha_2^{(3)}\sin\tau + \tfrac{1}{2}\beta\gamma_2^{(2)}\cos 2\tau - \tfrac{1}{2}\beta\gamma_1^{(2)}\sin 2\tau,$$
$$(63)\quad \psi_3 = [\tfrac{3}{2}\alpha_2^{(2)}\beta - \tfrac{3}{2}i\theta_1^2\beta - \tfrac{21}{16}i\beta^3] + \gamma_1^{(3)}\cos\tau + \gamma_2^{(3)}\sin\tau$$
$$+ [\tfrac{1}{2}\alpha_2^{(2)}\beta - \tfrac{11}{6}i\theta_1^2\beta - \tfrac{7}{16}i\beta^3]\cos 2\tau + [-\tfrac{1}{2}\alpha_1^{(2)}\beta - \tfrac{11}{6}\theta_1^2\beta + \tfrac{3}{16}\beta^3]\sin 2\tau.$$

It can be shown by induction at this point that ϕ and λ involve only even powers of μ, and that ψ is an odd series in μ. Furthermore ϕ contains only odd multiples of τ, and ψ only even multiples. Consequently

$$\gamma_1^{(2)} = \gamma_2^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = \gamma_1^{(3)} = \gamma_2^{(3)} = 0,$$

and all

$$\phi_{2j+1} = \psi_{2j} = 0.$$

Coefficient of μ^4.

$$\phi_4'' + \phi_4 = [-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2]\cos\tau + [2i\lambda_4$$
$$(64)\qquad + 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4]\sin\tau + [6\theta_1^2\beta^2 + 3\beta^2\alpha_1^{(2)} + \tfrac{3}{16}\beta^4]\cos 3\tau$$
$$+ [-6i\theta_1^2\beta^2 + 3\beta^2\alpha_2^{(2)} - \tfrac{45}{16}i\beta^4]\sin 3\tau - \tfrac{21}{16}\beta^4\cos 5\tau + \tfrac{21}{16}i\beta^4\sin 5\tau.$$

Therefore, from the periodicity condition, we must have

$$-2\lambda_4 + 4\theta_1^2(\alpha_1^{(2)} - i\alpha_2^{(2)}) - 16\theta_1^4 - 10\theta_1^2\beta^2 = 0,$$
$$2i\lambda_4 + 4\theta_1^2(i\alpha_1^{(2)} + \alpha_2^{(2)}) + 16i\theta_1^4 \qquad\qquad = 0.$$

Solving these equations, we find

$$\lambda_4 = -8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2, \qquad \alpha_1^{(2)} - i\alpha_2^{(2)} = \tfrac{5}{4}\beta^2.$$

In this last equation we can choose $\alpha_1^{(2)} = \frac{5}{4}\beta^2$ and $\alpha_2^{(2)} = 0$. This choice of $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ will make the coefficient of $\sin\tau$ in ϕ_2 equal to zero, and since the same thing occurs for each ϕ_j it is evident that this method of choosing will simplify the solution by making the coefficient of $\sin \tau$ equal to zero for all powers of μ. We have then on integrating

$$
(65)\quad \begin{aligned}
\phi_4 &= \alpha_4 \cos \tau + [-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4] \cos 3\tau + i[\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4] \sin 3\tau \\
&\quad + \tfrac{7}{128}\beta^4 \cos 5\tau - \tfrac{7}{128} i\beta^4 \sin 5\tau, \\
\psi_3 &= i[-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3] + i[-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3]\cos 2\tau + [-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3]\sin 2\tau, \\
\phi_2 &= \tfrac{5}{4}\beta^2 \cos \tau - \tfrac{3}{8}\beta^2 \cos 3\tau + \tfrac{3}{8} i\beta^2 \sin 3\tau, \\
\psi_1 &= -\tfrac{3}{2} i\beta - \tfrac{1}{2} i\beta \cos 2\tau - \tfrac{1}{2}\beta \sin 2\tau, \\
\phi_0 &= \cos \tau - i \sin \tau, \\
\lambda &= 2\theta_1^2 \mu^2 + (-8\theta_1^4 - \tfrac{5}{2}\theta_1^2\beta^2)\mu^4 + \cdots.
\end{aligned}
$$

The coefficient α_4 of $\cos \tau$ in ϕ_4 is determined by the periodicity condition on ϕ_6. That the process of determining the values of the λ_j and the constants of integration arising at each step is general may be shown as follows. Let us suppose that we have computed everything up to and including ϕ_j with the exception of the constants of integration in ϕ_j. We have then

$$\phi_j = \alpha_1^{(j)} \cos \tau + \alpha_2^{(j)} \sin \tau + \text{known terms}.$$

The $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$ enter ψ_{j+1} as follows:

$$
\begin{aligned}
\psi_{j+1}'' + \psi_{j+1} &= 3\beta \sin \tau \cdot \phi_j + \text{known terms}, \\
&= \tfrac{3}{2}\beta\alpha_2^{(j)} - \tfrac{3}{2}\beta\alpha_2^{(j)} \cos 2\tau + \tfrac{3}{2}\beta\alpha_1^{(j)} \sin 2\tau + \text{known terms}.
\end{aligned}
$$

Consequently in so far as it involves $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$

$$\psi_{j+1} = \tfrac{3}{2}\beta\alpha_2^{(j)} + \tfrac{1}{2}\beta\alpha_2^{(j)} \cos 2\tau - \tfrac{1}{2}\beta\alpha_1^{(j)} \sin 2\tau.$$

Similarly, in so far as ϕ_{j+2} depends upon constants as yet undetermined

$$
(66)\quad \begin{aligned}
\phi_{j+2}'' + \phi_{j+2} &= -2i\lambda_2\phi_j' - 2i\lambda_{j+2}\phi_0' + [(4\theta_1^2 - \tfrac{3}{2}\beta^2) + \tfrac{9}{2}\beta^2 \cos 2\tau]\phi_j + 3\beta \sin \tau \psi_{j+1} \\
&= [-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2}] \cos \tau \\
&\quad + [2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2}] \sin \tau \\
&\quad + 3\beta^2\alpha_1^{(j)} \cos 3\tau + 3\beta^2\alpha_2^{(j)} \sin 3\tau,
\end{aligned}
$$

where A_{j+2} and B_{j+2} are the known terms in the coefficients of $\cos \tau$ and $\sin \tau$ respectively. From the periodicity condition we must have

$$
(67)\quad \begin{aligned}
-2\lambda_{j+2} + 4\theta_1^2(\alpha_1^{(j)} - i\alpha_2^{(j)}) + A_{j+2} &= 0, \\
+2i\lambda_{j+2} + 4\theta_1^2(i\alpha_1^{(j)} + \alpha_2^{(j)}) + B_{j+2} &= 0.
\end{aligned}
$$

The solution of these equations is

$$(68)\qquad \begin{aligned} \lambda_{j+2} &= \tfrac{1}{4}(A_{j+2} + iB_{j+2}), \\ \alpha_1^{(1)} - i\alpha_2^{(j)} &= -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}. \end{aligned}$$

As has already been pointed out we can choose $\alpha_1^{(j)} = 0$ and we have then

$$(69)\qquad \alpha_2^{(j)} = -\frac{A_{j+2} - iB_{j+2}}{8\theta_1^2}.$$

In order to show that λ_{j+2} and $\alpha_1^{(j)}$ are real it will be sufficient to show that A_{j+2} is real and B_{j+2} is a pure imaginary. This is readily shown by induction, for up to $j = 4$ inclusive we have

$$\phi_j = \sum_\kappa m_\kappa \cos \kappa\tau + i \sum_\kappa n_\kappa \sin \kappa\tau,$$

$$\psi_j = i \sum_\kappa f_\kappa \cos \kappa\tau + \sum_\kappa g_\kappa \sin \kappa\tau,$$

where m_κ, n_κ, f_κ and g_κ are all real. From the form of the differential equations it follows at once that the same forms hold for $j = 5$, then $j = 6$, and so on. That is A_{j+2} is real while B_{j+2} is a pure imaginary.

It is further to be noticed that A_{j+2} and B_{j+2} do not contain any terms in β independent of θ_1^2, and consequently the θ_1^2 which appears in the denominator of $\alpha_1^{(j)}$ will divide out. This is proved as follows. If θ_1^2 be put equal to zero in the differential equations then equations (52) become the equations of variation of a circular orbit in the ordinary two-body problem, the plane of the circle being inclined to the plane of reference by an angle whose sine is $\beta\mu = s$. The original differential equations (6) can then be written

$$(70)\qquad r'' = \frac{(1-s^2)(1-e^2)}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} = R, \qquad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} = Q,$$

where the constant c^2 is given the form $(1-s^2)(1-e^2)$. For these equations we have the solution

$$(71)\qquad \begin{aligned} r &= a\frac{(1-e^2)\sqrt{1 - s^2\sin^2(\theta - \Omega)}}{1 + e\cos(\theta - \theta_0)}, \\ q &= a\frac{(1-e^2)s\sin(\theta - \Omega)}{1 + e\cos(\theta - \theta_0)}, \end{aligned}$$

where

$$(\theta - \theta_0) = (\tau - \tau_0) + 2e\sin(\tau - \tau_0) + \cdots.$$

Let us now form the equations of variation by putting

$$r = r_0 + \rho, \qquad q = q_0 + \sigma, \qquad e = e_0 + \epsilon,$$

where r_0, q_0 and e_0 are the values in (71). We find

$$(72)\qquad \begin{aligned} \rho'' &= \frac{\partial R}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial R}{\partial e}\epsilon, \\ \sigma'' &= \frac{\partial Q}{\partial r}\rho + \frac{\partial R}{\partial q}\sigma + \frac{\partial Q}{\partial e}\epsilon. \end{aligned}$$

Three solutions of these equations are given by*

$$(73)\qquad \begin{aligned} \rho &= c_1\frac{\partial r_0}{\partial\Omega}, & \rho &= c_2\frac{\partial r_0}{\partial\tau_0}, & \rho &= c_3\frac{\partial r_0}{\partial e_0}, \\ \sigma &= c_1\frac{\partial q_0}{\partial\Omega}, & \sigma &= c_2\frac{\partial q_0}{\partial\tau_0}, & \sigma &= c_3\frac{\partial q_0}{\partial e_0}, \\ \epsilon &= c_1\frac{\partial e_0}{\partial\Omega}, & \epsilon &= c_2\frac{\partial e_0}{\partial\tau_0}, & \epsilon &= c_3\frac{\partial e_0}{\partial e_0}. \end{aligned}$$

If $e_0 \neq 0$ these three solutions are distinct, but the case in which we are interested is when $e_0 = 0$. In this case it is not difficult to see that the first two solutions coincide. Since the equations are linear, the system

$$(74)\qquad \begin{aligned} \rho &= c_4\left[\frac{\partial r_0}{\partial\Omega} - \frac{\partial r_0}{\partial\tau_0}\right], \\ \sigma &= c_4\left[\frac{\partial q_0}{\partial\Omega} - \frac{\partial q_0}{\partial\tau_0}\right], \\ \epsilon &= c_4\left[\frac{\partial e_0}{\partial\Omega} - \frac{\partial e_0}{\partial\tau_0}\right], \end{aligned}$$

is also a solution, but as it vanishes for $e_0 = 0$ it carries e_0 as a factor. We can divide out this factor and absorb it into the arbitrary c_4. For $e_0 = 0$ this solution does not now vanish, and it is moreover distinct from the first solution. Thus we have three distinct solutions even when $e_0 = 0$, but since $\partial Q/\partial e_0 \equiv 0$, and $\partial R/\partial e_0$ carries e_0 as a factor, equations (72) pass over to the equations of variation of a circle when $e_0 = 0$. For these equations we have three solutions which are periodic with the period 2π. The fourth solution is not periodic but involves a term of the form τ times a periodic function.

Let us return now to the solution which we have developed, (65), and consider only the terms which belong to the two-body problem, viz., the terms which are

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 163.

independent of θ_1^2. This solution may be separated into two solutions, one of which is real, the other a pure imaginary. The real solution is the third solution of (73), and the purely imaginary is the second solution. Since both of these solutions are certainly periodic with the period 2π, it follows that no term in β alone can occur in the A_{j+2} and B_{j+2} of (66) for the presence of such terms would give rise to non-periodic terms in the two-body problem. Hence A_{j+2} and B_{j+2} carry θ_1^2 as a factor which may be divided out in equation (69). Further λ_{j+2}, (68), carries θ_1^2 as a factor and therefore λ vanishes with the oblateness of the spheroid.

The solution (65) may now be written

$$(75)\qquad \begin{aligned} \rho^{(1)} &= e^{i\lambda\tau}[\phi^{(1)} + i\phi^{(2)}], \\ \sigma^{(1)} &= e^{i\lambda\tau}[\psi^{(1)} + i\psi^{(2)}], \end{aligned}$$

where

$$(76)\qquad \begin{aligned} \phi^{(1)} &= \cos\tau + [\tfrac{5}{4}\beta^2\cos\tau - \tfrac{3}{8}\beta^2\cos 3\tau]\,\mu^2 \\ &\quad + [\alpha_4\cos\tau + (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{63}{128}\beta^4)\cos 3\tau + \tfrac{7}{128}\beta^4\cos 5\tau]\,\mu^4 + \cdots, \\ \phi^{(2)} &= -\sin\tau + [\tfrac{3}{8}\beta^2\sin 3\tau]\,\mu^2 \\ &\quad + [(\tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{45}{128}\beta^4)\sin 3\tau - \tfrac{7}{128}\beta^4\sin 5\tau]\,\mu^4 + \cdots, \\ \psi^{(1)} &= [-\tfrac{1}{2}\beta\sin 2\tau]\,\mu + [(-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\sin 2\tau]\,\mu^3 + \cdots, \\ \psi^{(2)} &= [-\tfrac{3}{2}\beta - \tfrac{1}{2}\beta\cos 2\tau]\,\mu \\ &\quad + [(-\tfrac{3}{2}\theta_1^2\beta - \tfrac{21}{16}\beta^3) + (-\tfrac{11}{6}\theta_1^2\beta - \tfrac{7}{16}\beta^3)\cos 2\tau]\,\mu^3 + \cdots. \end{aligned}$$

By putting

$$e^{i\lambda\tau} = \cos\lambda\tau + i\sin\lambda\tau$$

we can write

$$(77)\qquad \begin{aligned} \rho^{(1)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] + i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(1)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] + i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned}$$

We have thus one solution of the differential equations. A second solution can be derived from it by merely changing the sign of i. Thus

$$(78)\qquad \begin{aligned} \rho^{(2)} &= [\phi^{(1)}\cos\lambda\tau - \phi^{(2)}\sin\lambda\tau] - i[\phi^{(2)}\cos\lambda\tau + \phi^{(1)}\sin\lambda\tau], \\ \sigma^{(2)} &= [\psi^{(1)}\cos\lambda\tau - \psi^{(2)}\sin\lambda\tau] - i[\psi^{(2)}\cos\lambda\tau + \psi^{(1)}\sin\lambda\tau]. \end{aligned}$$

By adding and subtracting these two solutions we have finally

$$(79)\qquad \begin{aligned} \rho &= A[\rho^{(1)} + \rho^{(2)}] + B[\rho^{(1)} - \rho^{(2)}], \\ \sigma &= A[\sigma^{(1)} + \sigma^{(2)}] + B[\sigma^{(1)} - \sigma^{(2)}], \end{aligned}$$

A and B being two arbitrary constants. As above developed there is a certain arbitrariness in these solutions owing to the manner in which the constants of integration were determined. They may be reduced to a normal form by multiplying each solution by the proper power series in μ^2 with constant coefficients. By this process we can make for $\tau = 0$,

$$\rho^{(1)}(0) + \rho^{(2)}(0) = 1, \qquad \sigma^{(1)}(0) - \sigma^{(2)}(0) = \beta\mu.$$

Since $[\rho^{(1)} - \rho^{(2)}]$ and $[\sigma^{(1)} + \sigma^{(2)}]$ are sine series they vanish for $\tau = 0$.

The third and fourth solutions of the equations of variation, (51), are given by *

$$(80) \qquad \begin{aligned} \rho_3 &= C\frac{\partial r_0}{\partial \tau_0}, & \rho_4 &= D'\frac{\partial(ar_0)}{\partial a}, \\ \sigma_3 &= C\frac{\partial q_0}{\partial \tau_0}, & \sigma_4 &= D'\frac{\partial(aq_0)}{\partial a}. \end{aligned}$$

In forming the partial derivatives with respect to a in the fourth solution it should be remembered that θ_1^2 is an explicit function of a, by equations (4), and that τ is also a function of a implicitly through n. The third solution also can be normalized by giving the arbitrary constant such a form that at $\tau = 0$

$$\rho_3 = 0, \qquad \sigma_3 = C\beta\mu.$$

The fourth solution is non-periodic and has the form

$$(81) \qquad \rho = D\left[\tau\frac{\partial r_0}{\partial \tau_0} + \phi_4\right], \qquad \sigma = D\left[\tau\frac{\partial q_0}{\partial \tau_0} + \psi_4\right],$$

where ϕ_4 and ψ_4 are periodic functions of τ with the period 2π. As in the previous solutions this may be normalized so that at $\tau = 0$

$$\rho_4 = D\beta^2\mu^2, \qquad \sigma_4 = 0.$$

It is also easy to find ϕ_4 and ψ_4 by substituting (81) in the equations of variation and solving for these variables, which must be periodic.

Carrying out the above operations we find the following general solutions

$$\begin{aligned} \rho = A\{&\cos(1-\lambda)\tau + [-\tfrac{1}{4}\beta^2\cos(1-\lambda)\tau + \tfrac{5}{8}\beta^2\cos(1+\lambda)\tau - \tfrac{3}{8}\beta^2\cos(3-\lambda)\tau]\mu^2 \\ &+ [(-\tfrac{1}{2}\alpha_4 + \tfrac{3}{4}\theta_1^2\beta^2 + \tfrac{21}{32}\beta^4)\cos(1-\lambda)\tau + (\tfrac{1}{2}\alpha_4 - \tfrac{35}{64}\beta^4)\cos(1+\lambda)\tau \\ &+ (-\tfrac{3}{4}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4)\cos(3-\lambda)\tau - \tfrac{9}{128}\beta^4\cos(3+\lambda)\tau \\ &+ \tfrac{7}{128}\beta^4\cos(5-\lambda)\tau]\mu^4 + \cdots\} \end{aligned}$$

* See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 163.

$$
\begin{aligned}
(82)\qquad & + B\{\tfrac{1}{2}\sin(1-\lambda)\tau + [(-\tfrac{1}{8}\beta^2 - \tfrac{5}{6}\theta_1^2)\sin(1-\lambda)\tau - \tfrac{5}{16}\beta^2\sin(1+\lambda)\tau \\
& \qquad - \tfrac{3}{16}\beta^2\sin(3-\lambda)\tau]\mu^2 \cdots\} \\
& + C\{[-\tfrac{1}{2}\beta^2\sin 2\tau]\mu^2 + [(\tfrac{1}{16}\theta_1^2\beta^2 - \tfrac{1}{8}\beta^4)\sin 2\tau + \tfrac{1}{16}\beta^4\sin 4\tau]\mu^4 + \cdots\} \\
& + D\Big\{\{[\tfrac{3}{4}\beta^2 + \tfrac{1}{4}\beta^2\cos 2\tau]\mu^2 + [(\tfrac{9}{8}\theta_1^2\beta^2 - \tfrac{3}{32}\beta^4) \\
& \qquad + (-\tfrac{9}{8}\theta_1^2\beta^2 + \tfrac{1}{8}\beta^4)\cos 2\tau - \tfrac{1}{32}\beta^4\cos 4\tau]\mu^4 \cdots\} + \tau\{[\tfrac{3}{4}\beta^4\sin 2\tau]\mu^4 \\
& \qquad + [(\tfrac{91}{16}\theta_1^2\beta^4 + \tfrac{15}{64}\beta^6)\sin 2\tau - \tfrac{3}{32}\beta^6\sin 4\tau]\mu^6 + \cdots\}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
& \sigma = A\{[\tfrac{3}{2}\beta\sin\lambda\tau - \tfrac{1}{2}\beta\sin(2-\lambda)\tau]\mu + [\tfrac{3}{2}\theta_1^2\beta\sin\lambda\tau \\
& \qquad - \tfrac{11}{6}\theta_1^2\beta\sin(2-\lambda)\tau]\mu^3 + \cdots\} \\
& \quad + B\{[\tfrac{3}{4}\beta\cos\lambda\tau + \tfrac{1}{4}\beta\cos(2-\lambda)\tau]\mu + [-\tfrac{1}{2}\theta_1^2\beta\cos\lambda\tau \\
(83)\qquad & \qquad + \tfrac{1}{2}\theta_1^2\beta\cos(2-\lambda)\tau]\mu^3 + \cdots\} \\
& \quad + C\{[\beta\cos\tau]\mu + [0]\mu^3 + \cdots\} \\
& \quad + D\Big\{\{[\tfrac{1}{2}\beta\sin\tau]\mu + [(-\tfrac{7}{4}\theta_1^2\beta + \tfrac{1}{2}\beta^3)\sin\tau]\mu^3 + \cdots\} \\
& \qquad + \tau\{[-\tfrac{3}{2}\beta^3\cos\tau]\mu^3 + [(-\tfrac{53}{16}\theta_1^2\beta^3 + \tfrac{21}{16}\beta^5)\cos\tau]\mu^5 + \cdots\}\Big\}.
\end{aligned}
$$

§ 13. *Non-homogeneous linear differential equations with periodic coefficients.*

If we have a set of homogeneous linear differential equations

$$
(84)\qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij}(t)x_j, \qquad (i = 1, \cdots, n),
$$

where the θ_{ij} are periodic functions of t with the period 2π, we know from the writings of Floquet * that, if the roots of the fundamental equation are all distinct, the solution has the form

$$
(85)\qquad x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t),
$$

where the A_j are arbitrary constants, the α_j are constants known as the characteristic exponents, and the ϕ_{ij} are periodic functions of t with the period 2π.

Such equations arise in dynamics whenever we study small variations from a known periodic solution. If we confine our attention entirely to the first powers of the variations the equations are linear and homogeneous, and are known as the "equations of variation." They have been studied extensively by Poincaré in *Les Méthodes Nouvelles de la Mécanique Céleste.* If the second and higher

* Annales Scientifiques de l'École Normale Supérieure, 2d series, vol. 12 (1883), p. 47.

powers of the variation are considered equations of the same type arise but they are no longer homogeneous. It is necessary for our purpose to derive the character of the solutions of such equations when the non-homogeneous terms are periodic, even though the period be different from 2π. We will suppose that the equations are the same as (84) with the addition of periodic terms.

Case I.

We will assume first that the period of the non-homogenous terms is 2π. The equations are then

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + g_i(t) \qquad (i=1, \cdots, n), \tag{86}$$

where $\theta_{ij}(t)$ and $g_i(t)$ are periodic with the period 2π. The solutions of (86) may be written

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t} \phi_{ij}(t) + \psi_i(t) \qquad (i=1, \cdots, n). \tag{87}$$

If in the differential equations we change t into $t + 2\pi$ the equations remain unchanged, but the solutions become

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j (t+2\pi)} \phi_{ij}(t + 2\pi) + \psi_i(t + 2\pi). \tag{88}$$

Therefcre equations (88) are also solutions of (86), and consequently

$$\frac{d}{dt} \psi_i(t + 2\pi) = \sum_{j=1}^{n} \theta_{ij} \psi_j(t + 2\pi) + g_i(t);$$

also

$$\frac{d}{dt} \psi_i(t) = \sum_{j=1}^{n} \theta_{ij} \psi_j(t) + g_i(t).$$

Forming the difference of these equations we find

$$\frac{d}{dt} [\psi_i(t + 2\pi) - \psi_i(t)] = \sum_{j=1}^{n} \theta_{ij} [\psi_j(t + 2\pi) - \psi_j(t)] \quad (i=1, \cdots, n). \tag{89}$$

These equations are the same as (84) and their solutions have the same form as (85), that is

$$\psi_i(t + 2\pi) - \psi_i(t) = \sum_{j=1}^{n} B_j e^{\alpha_j t} \phi_{ij}(t). \tag{90}$$

The constants B_j in this case are not arbitrary but depend upon the differential equations. They may or may not be zero. Equations (90) may be interpreted as showing that $\psi_i(t)$ is composed of two parts, a periodic part and a non-periodic part. We may express it therefore in the form

(91) $$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j e^{\alpha_j t}\phi_{ij}(t) f_{ij}(t),$$

where $\omega_i(t)$ are periodic with the period 2π, and the $f_{ij}(t)$ are functions which must be determined. Changing t into $t + 2\pi$ in equations (91) and forming the difference $\psi_i(t + 2\pi) - \psi_i(t)$, we find

(92) $$\begin{aligned}\psi_i(t + 2\pi) - \psi_i(t) &= \sum_{j=1}^{n} B_j e^{\alpha_j t}\phi_{ij}(t)\left[e^{2\alpha_j\pi} f_{ij}(t + 2\pi) - f_{ij}(t)\right],\\ &= \sum_{j=1}^{n} B_j e^{\alpha_j t}\phi_{ij}(t) \quad \text{from equations (90).}\end{aligned}$$

Comparing the coefficients in these equalities we see that

(93) $$e^{2\alpha_j\pi} f_{ij}(t + 2\pi) - f_{ij}(t) = 1,$$

from which we can determine the character of the functions $f_{ij}(t)$.

For this purpose let us define a new function $\lambda_{ij}(t)$ such that

(94) $$\lambda_{ij}(t) = e^{\alpha_j t} f_{ij}(t) + \frac{e^{\alpha_j t}}{1 - e^{2\alpha_j\pi}}.$$

Then by virtue of the relations (93) the $\lambda_{ij}(t)$ are periodic with the period 2π, for we have

$$\begin{aligned}\lambda_{ij}(t + 2\pi) &= e^{\alpha_j t}\cdot e^{2\alpha_j\pi} f_{ij}(t + 2\pi) + \frac{e^{\alpha_j t}\cdot e^{2\alpha_j\pi}}{1 - e^{2\alpha_j\pi}},\\ &= e^{\alpha_j t}\left[f_{ij}(t) + 1\right] + \frac{e^{\alpha_j t}\cdot e^{2\alpha_j\pi}}{1 - e^{2\alpha_j\pi}},\\ &= e^{\alpha_j t} f_{ij}(t) + \frac{e^{\alpha_j t}}{1 - e^{2\alpha_j\pi}},\\ &= \lambda_{ij}(t).\end{aligned}$$

On solving (94) we find

(95) $$f_{ij}(t) = e^{-\alpha_j t}\lambda_{ij}(t) + \frac{1}{e^{2\alpha_j\pi} - 1},$$

where the λ_{ij} are periodic with the period 2π. This expression for the $f_{ij}(t)$ substituted in (91) gives

(96) $$\psi_i(t) = \omega_i(t) + \sum_{j=1}^{n} B_j\phi_{ij}\lambda_{ij} + \sum_{j=1}^{n}\frac{B_j}{e^{2\alpha_j\pi} - 1} e^{\alpha_j t}\phi_{ij}(t).$$

The terms included under the last summation sign are merely terms of the complementary function. All the other terms are periodic with the period 2π.

This form for the $\psi_i(t)$ fails however if for any j

$$e^{2\alpha_j\pi} = 1,$$

that is, if

$$\alpha_j \equiv 0 \mod \sqrt{-1}.$$

In this event equations (93) for such values of j become

$$f_{ij}(t + 2\pi) - f_{ij}(t) = 1. \tag{97}$$

We will define the corresponding value of λ_{ij} by the relation

$$\lambda_{ij}(t) = f_{ij}(t) - \frac{t}{2\pi}. \tag{98}$$

By virtue of (97) λ_{ij} is periodic with the period 2π, for

$$\begin{aligned} \lambda_{ij}(t + 2\pi) &= f_{ij}(t + 2\pi) - \frac{t + 2\pi}{2\pi}, \\ &= f_{ij}(t) - \frac{t}{2\pi}, \\ &= \lambda_{ij}(t). \end{aligned}$$

Solving (98) for $f_{ij}(t)$, we obtain

$$f_{ij}(t) = \lambda_{ij}(t) + \frac{t}{2\pi}. \tag{99}$$

Substituting this expression in (91), we find

$$\psi_i(t) = \text{periodic terms} + \frac{t}{2\pi}\sum_j B_j e^{\alpha_j t}\phi_{ij}(t), \tag{100}$$

the summation in the last term to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

We have then the general expression for the solution

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t}\phi_{ij}(t) + \text{periodic terms} + \frac{t}{2\pi}\sum_{j} B_j e^{\alpha_j t}\phi_{ij}(t), \tag{101}$$

where the summation in the last term is to be extended over all j such that $\alpha_j \equiv 0 \mod \sqrt{-1}$.

This result may be stated thus:

THEOREM I. *If the* $\theta_{ij}(t)$ *and* $g_i(t)$ *are periodic with the period* 2π *and if the characteristic exponents* α_j *are distinct and none of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution is periodic with the period* 2π. *If the* α_j *are distinct but some of them congruent to zero* mod $\sqrt{-1}$, *then the particular solution may involve terms of the form t times the corresponding complementary function.*

Case II.

We will suppose that the $g_i(t)$ are periodic and that the period is different from 2π. We will suppose further that $g_i(t)$ are expressible in the form

$$(102) \quad g_i(t) = \sum_j \left[a_{ij} \cos (j+\beta)t + b_{ij} \sin (j+\beta)t\right] = e^{\sqrt{-1}\beta t} f_i^{(1)}(t) + e^{-\sqrt{-1}\beta t} f_i^{(2)}(t),$$

where $f_i^{(1)}(t)$ and $f_i^{(2)}(t)$ are periodic with the period 2π and β is any real number not an integer. The differential equations may then be written

$$(103) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + e^{\sqrt{-1}\beta t} f_i^{(1)}(t) + e^{-\sqrt{-1}\beta t} f_i^{(2)}(t).$$

Since the equations are linear we can consider the particular solutions depending upon $e^{\sqrt{-1}\beta t}$ and $e^{-\sqrt{-1}\beta t}$ separately. The complete solution will be the sum of the two. Let us consider first

$$(104) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + e^{\sqrt{-1}\beta t} f_i^{(1)}(t).$$

If we make the substitution

$$x_i = e^{\sqrt{-1}\beta t} y_i$$

equations (104) become after dividing out the exponential

$$(105) \qquad \frac{dy_i}{dt} + \sqrt{-1}\, y_i = \sum_{j=1}^{n} \theta_{ij} y_j + f_i^{(1)}(t).$$

Since θ_{ij} and $f_i^{(1)}$ are periodic with the period 2π the discussion of the character of the y_i reduces to Case I. The characteristic exponents of equations (105) are $(\alpha_j - \sqrt{-1}\,\beta)$, where the α_j are the characteristic exponents for the homogeneous equations (84). Applying Theorem I we conclude that the y_i are periodic with the period 2π provided

$$\alpha_j \not\equiv \sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1, \cdots, n).$$

If, however, for any value of j

$$\alpha_j \equiv \sqrt{-1}\beta \mod \sqrt{-1}$$

then, in general, non-periodic terms will arise just as in Case 1.

The discussion for the second part of the differential equations

$$(106) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + e^{-\sqrt{-1}\beta t} f_i^{(2)}(t)$$

shows similarly that the particular integral is periodic if

$$\alpha_j \not\equiv -\sqrt{-1}\beta \qquad (j=1, \cdots, n).$$

In equations arising in dynamics the characteristic exponents α_j enter in pairs, equal but of opposite sign. Consequently for such equations this last condition

$$\alpha_j \not\equiv -\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1, \cdots, n),$$

does not differ from

$$\alpha_j \not\equiv +\sqrt{-1}\beta \mod \sqrt{-1} \qquad (j=1, \cdots, n).$$

We have then the following

THEOREM II. *If the θ_{ij} are periodic with the period 2π, and if the $g_i(t)$ have the form*

$$g_i(t) = \sum_k \left[a_{ik} \cos (k + \beta)t + b_{ik} \sin (k + \beta)t \right],$$

and if the characteristic exponents α_j are distinct and none of them congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution has the form

$$x_i = \sum_m \left[c_{im} \cos (m + \beta)t + d_{im} \sin (m + \beta)t \right].$$

If any of the α_j are congruent to $\pm\sqrt{-1}\beta \mod \sqrt{-1}$, then the particular solution will, in general, contain, in addition to periodic terms, terms of the form t times the corresponding complementary function.

Let us suppose that one of the α_j is congruent to $\sqrt{-1}\beta \mod \sqrt{-1}$. Then, in general, the solution will contain non-periodic terms. If now the $g_i(t)$ contain also terms of the form $\frac{\sin}{\cos}(p + q\beta)t$, where p and q are integers, these terms will not, in general, give rise to non-periodic terms in the solution, for the α_j will not, in general, be congruent to $\sqrt{-1}\,q\beta$. Indeed, if β is not rational, for no integral value of q except unity will $\sqrt{-1}\,q\beta \equiv \alpha_j$. But if β (and therefore α_j also) is rational, then certain values of q do exist for which $\alpha_j \equiv \sqrt{-1}\,q\beta$. Let us set

$$\alpha = \frac{i}{j}\sqrt{-1}, \qquad \beta = \frac{I}{J},$$

where i and j are integers relatively prime, and similarly for I and J. By hypothesis

$$\frac{i}{j} - \frac{I}{J} \equiv 0 \mod 1;$$

then

$$\frac{i}{j} - q\frac{I}{J} \not\equiv 0 \mod 1$$

unless $q = 1 + lJ$ (l an integer). In this event a term of the form

${}^{\cos}_{\sin}(p + q\beta)t$ can also be written ${}^{\cos}_{\sin}(r + \beta)t$ (r an integer) which involves only the first multiple of β, and the solution in general involves non-periodic terms.

Case III.

When two of the characteristic exponents of the homogeneous set are equal the preceding arguments are not applicable. Let us suppose that the homogeneous equations are

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j \qquad (i = 1, \cdots, n), \tag{107}$$

but that $\alpha_{n-1} = \alpha_n$. The solution then has in general the form

$$x_i = \sum_{j=1}^{n-2} C_j e^{\alpha_j t} \phi_{ij}(t) + [A + Bt] e^{\alpha_n t} \phi_{in}(t) + B e^{\alpha_n t} \phi_{i(n-1)}(t).$$

Suppose now the given equations are

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij} x_j + g_i(t),$$

where the θ_{ij} are the same as in (107) and the g_i are periodic functions of t with the period 2π. Let the particular solutions which depend on the $g_i(t)$ be

$$x_i = \psi_i(t).$$

Just as in Case I we find

$$\frac{d}{dt}[\psi_i(t + 2\pi) - \psi_i(t)] = \sum_{j=1}^{n} \theta_{ij} [\psi_j(t + 2\pi) - \psi_j(t)], \tag{108}$$

and consequently

$$\psi_i(t + 2\pi) - \psi_i(t) = \sum_{j=1}^{n-2} C'_j e^{\alpha_j t} \phi_{ij}(t) + [A' + B't] e^{\alpha_n t} \phi_{in}(t) + B' e^{\alpha_n t} \phi_{i(n-1)}(t). \tag{109}$$

The terms included under the summation sign are obviously the same as those treated in Case I. Neglecting these we may write

$$\psi_i(t) = e^{\alpha_n t} \phi_{in}(t) \cdot f_{in}(t) + e^{\alpha_n t} \cdot \phi_{i(n-1)}(t) f_{i(n-1)}(t) + \text{periodic terms}, \tag{110}$$

where $f_{in}(t)$ and $f_{i(n-1)}(t)$ are functions whose forms are to be determined. Let us form the difference $\psi_i(t + 2\pi) - \psi_i(t)$ by means of (110) and compare the result with (109). From (110) we get

$$\begin{aligned} \psi_i(t + 2\pi) - \psi_i(t) = {} & [e^{2\alpha_n \pi} f_{in}(t + 2\pi) - f_{in}(t)]\, e^{\alpha_n t} \phi_{in}(t) \\ & + [e^{2\alpha_n \pi} f_{i(n-1)}(t) - f_{i(n-1)}(t)]\, e^{\alpha_n t} \phi_{i(n-1)}(t). \end{aligned} \tag{111}$$

Comparing this with (109) we see that we must have

$$(112)\qquad \begin{aligned} e^{2a_n\pi}f_{in}(t+2\pi)-f_{in}(t)&=A'+B't,\\ e^{2a_n\pi}f_{i(n-1)}(t+2\pi)-f_{i(n-1)}(t)&=B'. \end{aligned}$$

Let us now define two new functions

$$(113)\qquad \begin{aligned} \lambda_{in}(t)&=e^{a_nt}f_{in}(t)+\left[\frac{A'}{1-e^{2a_n\pi}}+\frac{2\pi B'e^{2a_n\pi}}{(1-e^{2a_n\pi})^2}\right]e^{a_nt}+\frac{B'te^{a_nt}}{1-e^{2a_n\pi}},\\ \lambda_{i(n-1)}(t)&=e^{a_nt}f_{i(n-1)}(t)+\frac{B'e^{a_nt}}{1-e^{2a_n\pi}}. \end{aligned}$$

By virtue of the relations (112), $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π, and consequently

$$(114)\qquad \begin{aligned} f_{in}(t)&=e^{-a_nt}\lambda_{in}(t)+\left[\frac{A'}{e^{2a_n\pi}-1}-\frac{2\pi B'e^{2a_n\pi}}{(e^{2a_n\pi}-1)^2}\right]+\frac{B't}{e^{2a_n\pi}-1},\\ f_{i(n-1)}(t)&=e^{-a_nt}\lambda_{i(n-1)}+\frac{B'}{e^{2a_n\pi}-1}, \end{aligned}$$

where $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic with the period 2π. Substituting these expressions in (110), we find

$$(115)\qquad \begin{aligned} \psi_i(t)=&\left[\frac{A'}{(e^{2a_n\pi}-1)}-\frac{2\pi B'e^{2a_n\pi}}{(e^{2a_n\pi}-1)^2}+\frac{B't}{(e^{2a_n\pi}-1)}\right]e^{a_nt}\phi_{in}(t)\\ &+\frac{B'}{(e^{2a_n\pi}-1)}e^{a_nt}\phi_{i(n-1)}(t)+\text{periodic terms.} \end{aligned}$$

Comparing these terms with (109) we see that they are merely terms of the complementary function and that therefore

$$x_i=\text{complementary function}+\text{periodic terms.}$$

These results hold provided $\alpha_n \not\equiv 0 \bmod \sqrt{-1}$. In the event $\alpha_n\equiv 0 \bmod \sqrt{-1}$ equations (112) become

$$(116)\qquad f_{in}(t+2\pi)-f_{in}(t)=A'+B't,\qquad f_{i(n-1)}(t+2\pi)-f_{i(n-1)}(t)=B'.$$

It is necessary to give new definitions to the λ functions. We will let

$$(117)\qquad \lambda_{in}(t)=f_{in}(t)-\frac{B'}{4\pi}t^2-\frac{A'-B'\pi}{2\pi}t,\qquad \lambda_{i(n-1)}(t)=f_{i(n-1)}(t)-\frac{B'}{2\pi}t,$$

which give

$$(118)\qquad f_{in}(t)=\lambda_{in}(t)+\frac{B'}{4\pi}t^2+\frac{A'-B'\pi}{2\pi}t,\qquad f_{i(n-1)}(t)=\lambda_{i(n-1)}(t)+\frac{B'}{2\pi}t.$$

It is readily verified that $\lambda_{in}(t)$ and $\lambda_{i(n-1)}(t)$ are periodic by virtue of (116). Substituting these expressions in (110), we find

$$(119)\quad \psi_i(t)=e^{\alpha_n t}\left\{\left[\frac{B'}{4\pi}t^2+\frac{A'-B'\pi}{2\pi}t\right]\phi_{in}(t)+\frac{B'}{2\pi}t\phi_{i(n-1)}(t)\right\}+\text{periodic terms.}$$

If we let

$$\frac{A'-B'\pi}{2\pi}=C,\qquad \frac{B'}{2\pi}=D,$$

we get

$$\psi_i(t)=e^{\alpha_n t}\{[Ct+\tfrac{1}{2}Dt^2]\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}+\text{periodic terms.}$$

This expression is not quite the same as t times the corresponding complementary function for in one term we have the coefficient $\frac{1}{2}D$ instead of D. The theorem for this case then is

THEOREM III. *If θ_{ij} and $g_i(t)$ are periodic with the period 2π, and if two of the α_j are equal but not congruent to zero* mod $\sqrt{-1}$, *then the particular solution consists of terms periodic with the period 2π plus a constant times the corresponding complementary function. But if two of the α_j are equal and congruent to zero so that the corresponding part of the complementary function has the form*

$$e^{\alpha_n t}\{(A+Bt)\phi_{in}(t)+B\phi_{i(n-1)}(t)\},$$

then the particular solution consists of a periodic function plus a term of the form

$$e^{\alpha_n t}\{(Ct+\tfrac{1}{2}Dt^2)\phi_{in}(t)+Dt\phi_{i(n-1)}(t)\}.$$

§ 14. *Special theorems for the equations of variation.*

The foregoing theorems presuppose merely the conditions that the coefficients are periodic with the period 2π. Further facts with regard to the solutions may be established when further facts are specified with regard to the coefficients of the differential equations. Our equations of variation (51) may be written

$$(120)\quad \frac{d\rho_1}{d\tau}=\rho_2,\qquad \frac{d\rho_2}{d\tau}=\bar{\bar{\theta}}_2\rho_1+\bar{\theta}_3\sigma_1,\qquad \frac{d\sigma_1}{d\tau}=\sigma_2,\qquad \frac{d\sigma_2}{d\tau}=\bar{\theta}_3\rho_1+\bar{\bar{\theta}}_4\sigma_1,$$

where the notation *with respect to the θ's* has the following significance: even subscripts denote functions even in τ, and odd subscripts denote functions odd in τ; one dash indicates that only odd multiples of τ are involved, and two dashes indicate that only even multiples of τ are involved. The solutions, equations (82) and (83), may be characterized in the same manner, and are then

$$
(121)\qquad
\begin{aligned}
\rho_1 &= A\bar{\alpha}_2(\tau) + B\bar{\alpha}_1(\tau) + C\bar{\bar{\alpha}}_3(\tau) + D[\bar{\bar{\alpha}}_4(\tau) + \tau\bar{\bar{\alpha}}_3(\tau)],\\
\rho_2 &= A\bar{\beta}_1(\tau) + B\bar{\beta}_2(\tau) + C\bar{\bar{\beta}}_4(\tau) + D[\bar{\bar{\beta}}_3(\tau) + \tau\bar{\bar{\beta}}_4(\tau)],\\
\sigma_1 &= A\bar{\gamma}_1(\tau) + B\bar{\gamma}_2(\tau) + C\bar{\gamma}_4(\tau) + D[\bar{\gamma}_3(\tau) + \tau\bar{\gamma}_4(\tau)],\\
\sigma_2 &= A\bar{\bar{\delta}}_2(\tau) + B\bar{\bar{\delta}}_1(\tau) + C\delta_3(\tau) + D[\bar{\delta}_4(\tau) + \tau\bar{\delta}_3(\tau)],
\end{aligned}
$$

where the notation is the same as for the θ's with the exception that in the first two solutions every integral multiple of τ is increased by $\pm\lambda\tau$, e. g., $\cos(3+\lambda)\tau$. On these terms the dashes refer only to the integral part of the coefficients of τ.

Suppose now we have the following non-homogeneous equations:

$$
(122)\qquad
\begin{aligned}
\frac{d\rho_1}{d\tau} &= \rho_2, & \frac{d\rho_2}{d\tau} &= \bar{\bar{\theta}}_2\rho_1 + \bar{\theta}_3\sigma_1 + g(\tau),\\
\frac{d\sigma_1}{d\tau} &= \sigma_2, & \frac{d\sigma_2}{d\tau} &= \bar{\theta}_3\rho_1 + \bar{\bar{\theta}}_4\sigma_1 + f(\tau),
\end{aligned}
$$

where $g(\tau)$ and $f(\tau)$ are periodic with the period 2π. Since the characteristic exponents are $\sqrt{-1}\,\lambda$, $-\sqrt{-1}\,\lambda$, 0, 0, by Theorem III the solution has the form

$$
(123)\qquad
\begin{aligned}
\rho_1 &= (\rho_1) + \xi_1 = (\rho_1) + \omega_1(\tau) + a\tau\bar{\bar{\alpha}}_3 + b[\tfrac{1}{2}\tau^2\bar{\bar{\alpha}}_3 + \tau\bar{\bar{\alpha}}_4],\\
\rho_2 &= (\rho_2) + \xi_2 = (\rho_2) + \omega_2(\tau) + a\tau\bar{\bar{\beta}}_4 + b[\tfrac{1}{2}\tau^2\bar{\bar{\beta}}_4 + \tau\bar{\bar{\beta}}_3],\\
\sigma_1 &= (\sigma_1) + \eta_1 = (\sigma_1) + \omega_3(\tau) + a\tau\bar{\gamma}_4 + b[\tfrac{1}{2}\tau^2\bar{\gamma}_4 + \tau\gamma_3],\\
\sigma_2 &= (\sigma_1) + \eta_2 = (\sigma_2) + \omega_4(\tau) + a\tau\bar{\delta}_3 + b[\tfrac{1}{2}\tau^2\bar{\delta}_3 + \tau\bar{\delta}_4],
\end{aligned}
$$

where (ρ_i), (σ_i) indicate the complementary function and ξ_i and η_i are the particular integrals, of which the ω_i are the periodic parts. The a and b are constants which depend upon the differential equations. Since the ξ_i and η_i satisfy the differential equations we have

$$
(124)\qquad
\begin{aligned}
\frac{d}{d\tau}\xi_1(\tau) &= \xi_2(\tau), & \frac{d}{d\tau}\xi_2(\tau) &= \bar{\bar{\theta}}_2\xi_1(\tau) + \bar{\theta}_3\eta_1(\tau) + g(\tau),\\
\frac{d}{d\tau}\eta_1(\tau) &= \eta_2(\tau), & \frac{d}{d\tau}\eta_2(\tau) &= \bar{\theta}_3\xi_1(\tau) + \bar{\bar{\theta}}_4\eta_1(\tau) + f(\tau).
\end{aligned}
$$

Changing τ into $-\tau$ in these equations, we get

$$
(125)\qquad
\begin{aligned}
\frac{d}{d\tau}\xi_1(-\tau) &= -\xi_2(-\tau), & \frac{d}{d\tau}\xi_2(-\tau) &= -\bar{\bar{\theta}}_2\xi_1(-\tau) + \bar{\theta}_3\eta_1(-\tau) - g(-\tau),\\
\frac{d}{d\tau}\eta_1(-\tau) &= -\eta_2(-\tau), & \frac{d}{d\tau}\eta_2(-\tau) &= +\bar{\theta}_3\xi_1(-\tau) - \bar{\bar{\theta}}_4\eta_1(-\tau) - f(-\tau).
\end{aligned}
$$

If now we make the additional hypothesis that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, equations (124) and (125) may be combined into the following set

$$
(126)\quad
\begin{aligned}
&\frac{d}{d\tau}[\xi_1(\tau)-\xi_1(-\tau)]=[\xi_2(\tau)+\xi_2(-\tau)],\\
&\frac{d}{d\tau}[\xi_2(\tau)+\xi_2(-\tau)]=\bar{\bar{\theta}}_2[\xi_1(\tau)-\xi_1(-\tau)]+\bar{\theta}_3[\eta_1(\tau)+\eta_1(-\tau)],\\
&\frac{d}{d\tau}[\eta_1(\tau)+\eta_1(-\tau)]=[\eta_2(\tau)-\eta_2(-\tau)],\\
&\frac{d}{d\tau}[\eta_2(\tau)-\eta_2(-\tau)]=\theta_3[\xi_1(\tau)-\xi_1(-\tau)]+\bar{\bar{\theta}}_4[\eta_1(\tau)+\eta_1(-\tau)].
\end{aligned}
$$

These equations are the same as (120). Therefore

$$
(127)\quad
\begin{aligned}
\xi_1(\tau)-\xi_1(-\tau)&=A\bar{\alpha}_2(\tau)+B\bar{\alpha}_1(\tau)+C\bar{\bar{\alpha}}_3(\tau)+D[\bar{\bar{\alpha}}_4(\tau)+\tau\bar{\bar{\alpha}}_3(\tau)],\\
\xi_2(\tau)+\xi_2(-\tau)&=A\bar{\beta}_1(\tau)+B\bar{\beta}_2(\tau)+C\bar{\bar{\beta}}_4(\tau)+D[\bar{\bar{\beta}}_3(\tau)+\tau\bar{\bar{\beta}}_4(\tau)],\\
\eta_1(\tau)+\eta_1(-\tau)&=A\bar{\bar{\gamma}}_1(\tau)+B\bar{\bar{\gamma}}_2(\tau)+C\bar{\gamma}_4(\tau)+D[\bar{\gamma}_3(\tau)+\tau\bar{\gamma}_4(\tau)],\\
\eta_2(\tau)-\eta_2(-\tau)&=A\bar{\bar{\delta}}_2(\tau)+B\bar{\bar{\delta}}_1(\tau)+C\bar{\bar{\delta}}_3(\tau)+D[\bar{\delta}_4(\tau)+\tau\bar{\delta}_3(\tau)].
\end{aligned}
$$

Putting $\tau = 0$ in these equations we get from the first and the fourth

$$
(128)\qquad 0 = A\alpha_2(0) + D\alpha_4(0), \qquad 0 = A\delta_2(0) + D\delta_4(0).
$$

Either $A = D = 0$, or the determinant $\alpha_2(0)\delta_4(0) - \delta_2(0)\alpha_4(0) = 0$. But it is readily verified that the determinant is not zero. Therefore $A = D = 0$. If we suppose that $\xi_2(0) = \eta_1(0) = 0$ (we shall be interested only in such cases), it follows from the second and third equations of (127) that

$$
0 = B\bar{\beta}_2(0) + C\bar{\bar{\beta}}_4(0), \qquad 0 = B\bar{\bar{\gamma}}_2(0) + C\bar{\gamma}_4(0),
$$

and hence $B = C = 0$. Consequently

$$
(129)\quad
\begin{aligned}
&\xi_1(\tau) - \xi_1(-\tau) = 0, \qquad {}_2(\tau) + \xi_2(-\tau) = 0,\\
&\eta_1(\tau) + \eta_1(-\tau) = 0, \qquad \eta_2(\tau) - \eta_2(-\tau) = 0.
\end{aligned}
$$

We have then

THEOREM IV. *If $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function of τ, and if $\xi_2(0) = \eta_1(0) = 0$, then $\xi_1(\tau)$ and $\eta_2(\tau)$ are even functions of τ, and $\xi_2(\tau)$ and $\eta_1(\tau)$ are odd functions of τ.*

In the same way it can be shown that if $g(\tau)$ is odd and $f(\tau)$ is even, and if $\xi_1(0)=\eta_2(0)=0$, then ξ_1 and η_2 are odd and ξ_2 and η_1 are even.

Let us suppose now that $g(\tau)$ is periodic and contains only even multiples of τ, and that $f(\tau)$ is periodic and contains only odd multiples of τ. The general form of the solution will be the same as (123). ξ_1, ξ_2, η_1, and η_2 satisfy the differential equations

$$(130)\quad \begin{aligned} &\frac{d}{d\tau}\xi_1(\tau)=\xi_2(\tau), &&\frac{d}{d\tau}\xi_2(\tau)=\bar\theta_2\xi_1(\tau)+\bar\theta_3\eta_1(\tau)+\bar g(\tau),\\ &\frac{d}{d\tau}\eta_1(\tau)=\eta_2(\tau), &&\frac{d}{d\tau}\eta_2(\tau)=\bar\theta_3\xi_1(\tau)+\bar\theta_4\eta_1(\tau)+\bar f(\tau).\end{aligned}$$

Let us denote $\xi_i(\tau+\pi)$ by $\xi_i'(\tau)$ and $\eta_i(\tau+\pi)$ by $\eta_i'(\tau)$. Then by changing τ into $\tau+\pi$ in (130) we have

$$(131)\quad \begin{aligned} &\frac{d}{d\tau}\xi_1'(\tau)=\xi_2'(\tau), &&\frac{d}{d\tau}\xi_2'(\tau)=\bar\theta_2\xi_1'(\tau)-\bar\theta_3\eta_1'(\tau)+\bar g(\tau),\\ &\frac{d}{d\tau}\eta_1'(\tau)=\eta_2'(\tau), &&\frac{d}{d\tau}\eta_2'(\tau)=-\bar\theta_3\xi_1'(\tau)+\bar\theta_4\eta_1'(\tau)-\bar f(\tau).\end{aligned}$$

From (130) and (131) it follows that

$$(132)\quad \begin{aligned} &\frac{d}{d\tau}[\xi_1-\xi_1']=[\xi_2-\xi_2'], &&\frac{d}{d\tau}[\xi_2-\xi_2']=\bar\theta_2[\xi_1-\xi_1']+\bar\theta_3[\eta_1+\eta_1'],\\ &\frac{d}{d\tau}[\eta_1+\eta_1']=[\eta_2+\eta_2'], &&\frac{d}{d\tau}[\eta_2+\eta_2']=\bar\theta_3[\xi_1-\xi_1']+\bar\theta_4[\eta_1+\eta_1'].\end{aligned}$$

The solutions of these equations, which have the form of (120), are

$$(133)\quad \begin{aligned} \xi_1-\xi_1'&=A\bar\alpha_2(\tau)+B\alpha_1(\tau)+C\bar\alpha_3(\tau)+D[\bar\alpha_4(\tau)+\tau\bar\alpha_3(\tau)],\\ \xi_2-\xi_2'&=A\beta_1(\tau)+B\bar\beta_2(\tau)+C\bar\beta_4(\tau)+D[\bar\beta_3(\tau)+\tau\bar\beta_4(\tau)],\\ \eta_1+\eta_1'&=A\bar\gamma_1(\tau)+B\bar\gamma_2(\tau)+C\bar\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\bar\gamma_4(\tau)],\\ \eta_2+\eta_2'&=A\bar\delta_2(\tau)+B\bar\delta_1(\tau)+C\delta_3(\tau)+D[\delta_4(\tau)+\tau\delta_3(\tau)].\end{aligned}$$

Forming these expressions directly from (123), we get

$$(134)\quad \begin{aligned} \xi_1-\xi_1'&=\omega_1(\tau)-\omega_1(\tau+\pi)-[a\pi+\tfrac12 b\pi^2]\bar\alpha_3(\tau)-b\pi[\tau\bar\alpha_3(\tau)+\bar\alpha_4(\tau)],\\ \xi_2-\xi_2'&=\omega_2(\tau)-\omega_2(\tau+\pi)-[a\pi+\tfrac12 b\pi^2]\bar\beta_4(\tau)-b\pi[\tau\bar\beta_4(\tau)+\bar\beta_3(\tau)],\\ \eta_1+\eta_1'&=\omega_3(\tau)+\omega_3(\tau+\pi)-[a\pi+\tfrac12 b\pi^2]\bar\gamma_4(\tau)-b\pi[\tau\bar\gamma_4(\tau)+\bar\gamma_3(\tau)],\\ \eta_2+\eta_2'&=\omega_4(\tau)+\omega_4(\tau+\pi)-[a\pi+\tfrac12 b\pi^2]\delta_3(\tau)-b\pi[\tau\bar\delta_3(\tau)+\delta_4(\tau)].\end{aligned}$$

Comparing (133) and (134) we see that

$$A = B = 0, \qquad C = -\left[a\pi + \tfrac{1}{2}b\pi^2\right], \qquad D = -b\pi,$$

$$\omega_1(\tau) - \omega_1(\tau + \pi) = 0, \qquad \omega_3(\tau) + \omega_3(\tau + \pi) = 0,$$

$$\omega_2(\tau) - \omega_2(\tau + \pi) = 0, \qquad \omega_4(\tau) + \omega_4(\tau + \pi) = 0.$$

Therefore $\omega_1(\tau)$ and $\omega_2(\tau)$ contain only even multiples of τ while $\omega_3(\tau)$ and $\omega_4(\tau)$ contain only odd multiples, and by carrying this result into (123) we find

$$(135)\qquad \begin{aligned} \xi_1 &= \bar{\omega}_1(\tau) + a\tau\bar{\alpha}_3(\tau) + b\left[\tfrac{1}{2}\tau^2\bar{\alpha}_3(\tau) + \tau\bar{\alpha}_4(\tau)\right], \\ \xi_2 &= \bar{\omega}_2(\tau) + a\tau\bar{\beta}_4(\tau) + b\left[\tfrac{1}{2}\tau^2\bar{\beta}_4(\tau) + \tau\bar{\beta}_3(\tau)\right], \\ \eta_1 &= \bar{\omega}_3(\tau) + a\tau\bar{\gamma}_4(\tau) + b\left[\tfrac{1}{2}\tau^2\bar{\gamma}_4(\tau) + \tau\bar{\gamma}_3(\tau)\right], \\ \eta_2 &= \bar{\omega}_4(\tau) + a\tau\bar{\delta}_3(\tau) + b\left[\tfrac{1}{2}\tau^2\bar{\delta}_3(\tau) + \tau\bar{\delta}_4(\tau)\right], \end{aligned}$$

and hence we have

THEOREM V. *If $g(\tau)$ contains only even multiples of τ and $f(\tau)$ contains only odd multiples, then ξ_1 and ξ_2 contain only even multiples of τ and η_1 and η_2 contain only odd multiples.*

If in addition to the above hypotheses we suppose that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function, then ξ_1 and η_2 are even functions of τ and ξ_2 and η_1 are odd functions. Therefore $b = 0$. But if $g(\tau)$ is an odd function and $f(\tau)$ is an even function, then ξ_1 and η_2 are odd functions and ξ_2 and η_1 are even functions, so that in this event $a = 0$.

In the same manner as above we prove

THEOREM VI. *If $g(\tau)$ contains only odd multiples of τ and $f(\tau)$ contains only even multiples, then ξ_1 and ξ_2 contain only odd multiples of τ and η_1 and η_2 contain only even multiples. Furthermore ξ_1, ξ_2, η_1 and η_2 are periodic with the period 2π.*

If $g(\tau)$ is of the form $\sum_j m_j \cos(j \pm \lambda)\tau$ and $f(\tau)$ has the form $\sum_j n_j \sin(j \pm \lambda)\tau$ then, since $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, the form of the solution is, by Theorem II,

$$(136)\qquad \begin{aligned} \xi_1 &= \sum_k p_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k p_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\alpha}_1(\tau) + a^{(2)}\tau\bar{\alpha}_2(\tau), \\ \xi_2 &= \sum_k q_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k q_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\beta}_2(\tau) + a^{(2)}\tau\bar{\beta}_1(\tau), \\ \eta_1 &= \sum_k r_k^{(2)} \cos(k \pm \lambda)\tau + \sum_k r_k^{(1)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\gamma}_2(\tau) + a^{(2)}\tau\bar{\gamma}_1(\tau), \\ \eta_2 &= \sum_k s_k^{(1)} \cos(k \pm \lambda)\tau + \sum_k s_k^{(2)} \sin(k \pm \lambda)\tau + a^{(1)}\tau\bar{\delta}_1(\tau) + a^{(2)}\tau\bar{\delta}_2(\tau); \end{aligned}$$

but since $g(\tau)$ is an even function and $f(\tau)$ is an odd function, ξ_1 and η_2 are even functions and ξ_2 and η_1 are odd functions. Therefore all the coefficients in (136) which have the upper index (2) are zero. But if $g(\tau)$ were an odd function of τ and $f(\tau)$ an even function then all the coefficients in (136) which have the upper index (1) would be zero. Therefore

THEOREM VII. *If* $g(\tau)$ *is of the form* $\sum_j m_j \cos(j \pm \lambda)\tau$ *and* $f(\tau)$ *has the form* $\sum_j n_j \sin(j \pm \lambda)\tau$, *where* $\pm\sqrt{-1}\lambda$ *are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a p_a \cos(a \pm \lambda)\tau + A\tau\bar{\alpha}_1(\tau),$$

$$\xi_2 = \sum_b p_b \sin(b \pm \lambda)\tau + A\tau\bar{\beta}_2(\tau),$$

$$\eta_1 = \sum_c p_c \sin(c \pm \lambda)\tau + A\tau\bar{\gamma}_2(\tau),$$

$$\eta_2 = \sum_d p_d \cos(d \pm \lambda)\tau + A\tau\bar{\delta}_1(\tau).$$

Also,

THEOREM VIII. *If* $g(\tau)$ *has the form* $\sum_j m_j \sin(j \pm \lambda)\tau$ *and* $f(\tau)$ *has the form* $\sum_j n_j \cos(j \pm \lambda)\tau$, *where* $\pm\sqrt{-1}\lambda$ *are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\xi_1 = \sum_a r_a \sin(a \pm \lambda)\tau + B\tau\bar{\alpha}_2(\tau), \qquad \xi_2 = \sum_b r_b \cos(b \pm \lambda)\tau + B\tau\bar{\beta}_1(\tau),$$

$$\eta_1 = \sum_c r_c \cos(c \pm \lambda)\tau + B\tau\bar{\gamma}_1(\tau), \qquad \eta_2 = \sum_d r_d \sin(d \pm \lambda)\tau + B\tau\bar{\delta}_2(\tau).$$

It is understood that in the above two theorems a, b, c, d, and j are integers.

§ 15. *Integration of the complete differential equations* (50).

It will be convenient hereafter to use the following notation for the solutions of the equations of variation:

$$(137) \qquad \begin{aligned} \rho &= A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D[\alpha_4(\tau) + \tau\alpha_3(\tau)], \\ \sigma &= A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D[\gamma_3(\tau) + \tau\gamma_4(\tau)]. \end{aligned}$$

The $\alpha_j(\tau)$ and $\gamma_j(\tau)$ are characterized thus:

$\alpha_2(\tau)$	involves only terms of the form	$\cos[(2n+1) \pm \lambda]\tau$,
$\gamma_1(\tau)$	" " " " " "	$\sin[2n \pm \lambda]\tau$,
$\alpha_1(\tau)$	" " " " " "	$\sin[(2n+1) \pm \lambda]\tau$,
$\gamma_2(\tau)$	" " " " " "	$\cos[2n \pm \lambda]\tau$,
$\alpha_3(\tau)$	" " " " " "	$\sin[2n]\tau$,
$\gamma_4(\tau)$	" " " " " "	$\cos[2n+1]\tau$,
$\alpha_4(\tau)$	" " " " " "	$\cos[2n]\tau$,
$\gamma_3(\tau)$	" " " " " "	$\sin[2n+1]\tau$.

It will be convenient also to write the differential equations for ρ and σ as

$$(138)\quad \begin{aligned} \rho'' + \theta_2\rho + \theta_3\sigma &= \theta_{001}\epsilon + \theta_{101}\epsilon\rho + \theta_{200}\rho^2 + \theta_{110}\rho\sigma + \theta_{020}\sigma^2 + \cdots, \\ \sigma'' + \theta_4\sigma + \theta_3\rho &= \bar{\theta}_{200}\rho^2 + \bar{\theta}_{110}\rho\sigma + \bar{\theta}_{020}\sigma^2 + \cdots, \end{aligned}$$

where all the θ's are periodic with the period 2π, and θ_2 and θ_4 contain only cosines of even multiples of τ, and θ_3 contains only sines of odd multiples of τ. On the right side of the first equation the coefficients of terms carrying odd powers of σ contain only sines of odd multiples of τ. All the other coefficients contain only cosines of even multiples. In the second equation odd powers of σ have coefficients involving only cosines of even multiples. All other coefficients contain only sines of odd multiples.

The initial conditions are

$$\rho = \alpha, \qquad \sigma = 0, \qquad \rho' = 0, \qquad \sigma' = \delta.$$

We will integrate equations (138) as power series in α, δ and ϵ, τ entering into the coefficients. From Poincaré's extension of Cauchy's theorem we know that these series are convergent for any arbitrarily chosen interval for τ, $0 \leqq \tau \leqq T$, provided $|\alpha|$, $|\delta|$ and $|\epsilon|$ are sufficiently small. The equations of variation involve the period $2\pi/\lambda$. The solutions are not periodic unless λ is rational. Hence θ_1^2 must be chosen in advance so that λ is rational. We will suppose then that $\lambda = J/K$, where J and K are integers, relatively prime. Then three solutions of the equations of variation are periodic with the period $2K\pi$.

Since ρ and σ are expansible in powers of α, δ and ϵ, we may write

$$\rho = \rho_{100}\alpha + \rho_{010}\delta + \rho_{001}\epsilon + \rho_{200}\alpha^2 + \rho_{110}\alpha\delta + \rho_{020}\delta^2 + \rho_{101}\alpha\epsilon + \rho_{011}\delta\epsilon + \rho_{002}\epsilon^2 + \cdots,$$

$$\sigma = \sigma_{100}\alpha + \sigma_{010}\delta + \sigma_{001}\epsilon + \sigma_{200}\alpha^2 + \sigma_{110}\alpha\delta + \sigma_{020}\delta^2 + \sigma_{101}\alpha\epsilon + \sigma_{011}\rho\epsilon + \sigma_{002}\epsilon^2 + \cdots.$$

Substituting these expressions in the differential equations (138) and equating the coefficients of similar powers, we have:

Coefficient of α.

$$\rho_{100}'' + \theta_2\rho_{100} + \theta_3\sigma_{100} = 0,$$

$$\sigma_{100}'' + \theta_4\sigma_{100} + \theta_3\rho_{100} = 0.$$

The solution of these equations is

$$\rho_{100} = A\alpha_2(\tau) + B\alpha_1(\tau) + C\alpha_3(\tau) + D\,[\,\alpha_4(\tau) + \tau\alpha_3(\tau)],$$

$$\sigma_{100} = A\gamma_1(\tau) + B\gamma_2(\tau) + C\gamma_4(\tau) + D\,[\gamma_3(\tau) + \tau\gamma_4(\tau)].$$

In order to satisfy the initial conditions we must have, at $\tau = 0$,

$$\rho_{100} = 1, \qquad \sigma_{100} = 0, \qquad \rho_{100}' = 0, \qquad \sigma_{100}' = 0.$$

From these conditions we find

$$(139)\qquad \begin{aligned} &A\alpha_2(0) + D\alpha_4(0) = 1, \\ &B\alpha_1'(0) + C\alpha_3'(0) = 0, \\ &B\gamma_2(0) + C\gamma_4(0) = 0, \\ &A\gamma_1'(0) + D[\gamma_3'(0) + \gamma_4(0)] = 0. \end{aligned}$$

The solution of these conditional equations is

$$(140)\qquad \begin{aligned} &B = 0, \qquad A = \frac{\gamma_3'(0) + \gamma_4(0)}{\Delta} = A_{100}^{(1)}, \\ &C = 0, \qquad D = -\frac{\gamma_1'(0)}{\Delta} = A_{100}^{(2)}, \\ &\Delta = \alpha_2(0)[\gamma_3'(0) + \gamma_4(0)] - \alpha_4(0)\gamma_1'(0). \end{aligned}$$

Hence the solution is

$$(141)\qquad \begin{aligned} \rho_{100} &= A_{100}^{(1)}\alpha_2(\tau) + A_{100}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)], \\ \sigma_{100} &= A_{100}^{(1)}\gamma_1(\tau) + A_{100}^{(2)}[\gamma_3(\tau) + \tau\alpha_4(\tau)]. \end{aligned}$$

Coefficient of δ.

$$(142)\qquad \begin{aligned} \rho_{010}'' + \theta_2\rho_{010} + \theta_3\sigma_{010} &= 0, \\ \sigma_{010}'' + \theta_4\sigma_{010} + \theta_3\rho_{010} &= 0. \end{aligned}$$

These equations are the same as for the coefficient of α. From the initial conditions we must have at $\tau = 0$

$$\rho_{010} = 0, \qquad \sigma_{010} = 0, \qquad \rho_{010}' = 0, \qquad \sigma_{010}' = 1.$$

The solutions are

$$(143)\qquad \begin{aligned} \rho_{010} &= A_{010}^{(1)}\alpha_2(\tau) + A_{010}^{(2)}[\alpha_4(\tau) + \tau\alpha_3(\tau)], \\ \sigma_{010} &= A_{010}^{(1)}\gamma_1(\tau) + A_{010}^{(2)}[\gamma_3(\tau) + \tau\gamma_4(\tau)], \end{aligned}$$

where

$$A_{010}^{(1)} = -\frac{\alpha_4(0)}{\Delta}, \qquad A_{010}^{(2)} = \frac{\alpha_2(0)}{\Delta}.$$

Coefficient of ϵ.

$$(144)\qquad \begin{aligned} \rho_{001}'' + \theta_2\rho_{001} + \theta_3\sigma_{001} &= \theta_{001}, \\ \sigma_{001}'' + \theta_4\sigma_{001} + \theta_3\rho_{001} &= 0. \end{aligned}$$

The right member, θ_{001}, is a periodic function of τ with the period 2π. Furthermore it involves only cosines of even multiples of τ. Hence by Theorem V the solution has the form

$$
(145)\quad
\begin{aligned}
\rho_{001} &= A\alpha_2(\tau)+B\alpha_1(\tau)+C\alpha_3(\tau)+D[\alpha_4(\tau)+\tau\alpha_3(\tau)]+\alpha_5(\tau)+a\tau\alpha_3(\tau),\\
\sigma_{001} &= A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\gamma_4(\tau)]+\gamma_5(\tau)+a\tau\gamma_4(\tau),
\end{aligned}
$$

where a is a constant depending on θ_{001}, $\alpha_5(\tau)$ is a cosine series involving only even multiples of τ, and $\gamma_5(\tau)$ involves only sines of odd multiples of τ.

From the initial conditions we must have at $\tau = 0$

$$
\rho_{001} = 0, \qquad \sigma_{001} = 0, \qquad \rho'_{001} = 0, \qquad \sigma'_{001} = 0.
$$

Determining the constants of integration so as to satisfy these conditions we have the solution

$$
(146)\quad
\begin{aligned}
\rho_{001} &= A^{(1)}_{001}\alpha_2(\tau) + A^{(2)}_{001}[\alpha_4(\tau)+\tau\alpha_3(\tau)] + \alpha_6(\tau),\\
\sigma_{001} &= A^{(1)}_{001}\gamma_1(\tau) + A^{(2)}_{001}[\gamma_3(\tau)+\tau\gamma_4(\tau)] + \gamma_6(\tau),
\end{aligned}
$$

where

$$
A^{(1)}_{001} = \frac{1}{\Delta}[\alpha_4\beta'_6 - \alpha_6(\beta_3+\beta'_4)],
$$

$$
A^{(2)}_{001} = \frac{1}{\Delta}[\alpha_6\beta'_1 - \alpha_1\beta'_6],
$$

$$
\alpha_6(\tau) = \alpha_5(\tau) - a\alpha_4(\tau), \qquad \gamma_6(\tau) = \gamma_5(\tau) - a\gamma_3(\tau).
$$

It will be seen later that the value of $\alpha_6(\tau)$ for $\tau = 0$ plays an important rôle, so that it is necessary for us to verify that it does not vanish. By hypothesis $\alpha_5(\tau)$ is the periodic part of the particular solution for ρ in the differential equations (144). Let us put in these equations

$$
\rho_{001} = \phi + a\tau\alpha_3(\tau), \qquad \sigma_{001} = \psi + a\tau\gamma_4(\tau),
$$

where ϕ and ψ are the periodic parts of the particular solution. We find

$$
\begin{aligned}
&\phi'' + \theta_2\phi + \theta_3\psi = -2a\alpha'_3(\tau) + 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - \tfrac{3}{4}\beta^2\cos 2\tau]\mu^2 + \cdots,\\
&\psi'' + \theta_4\psi + \theta_3\phi = -2a\beta'_3(\tau);
\end{aligned}
$$

or,

$$
\begin{aligned}
&\phi'' + \theta_2\phi + \theta_3\psi = 1 + [(-3\theta_1^2 + \tfrac{3}{4}\beta^2) - 2(a_0 + \tfrac{3}{4})\beta^2\cos 2\tau]\mu^2,\\
&\psi'' + \theta_4\psi + \theta_3\psi = -2a_0\beta\sin\tau\mu + \cdots,
\end{aligned}
$$

where we have substituted

$$
a = a_0 + a_2\mu^2 + \cdots.
$$

The functions ϕ and ψ can be expanded as series of the form

$$\phi = \phi_0 + \phi_2\mu^2 + \cdots, \qquad \psi = \psi_1\mu + \psi_3\mu^3 + \cdots.$$

Then we get

$$\phi_0'' + \phi_0 = 1, \qquad \text{so that} \qquad \phi_0 = 1 + c_0 \cos\tau,$$

$$\psi_1'' + \psi_1 = (3 - 2a_0)\beta \sin\tau + \tfrac{3}{2}\beta c_0 \sin 2\tau.$$

Since ψ_1 is periodic we must have $a_0 = \frac{3}{2}$, and then

$$\psi_1 = c_1 \sin\tau - \tfrac{1}{2}\beta c_0 \sin 2\tau,$$

$$\phi_2'' + \phi_2 = -\tfrac{1}{4}c_0\beta^2 \cos\tau + \text{other terms}.$$

Since ϕ_2 is periodic we must have $c_0 = 0$. Therefore

$$\phi = \alpha_5(\tau) = 1 + \mu^2 P_1(\mu^2), \qquad a = \tfrac{3}{2} + \mu^2 P_2(\mu^2),$$

$$\alpha_4(\tau) = 0 + [\tfrac{3}{2}\beta^2 + \tfrac{1}{2}\beta^2\cos 2\tau]\,\mu^2 + \cdots.$$

Consequently

(149) $$\alpha_6(0) = \alpha_5(0) - a\alpha_4(0) = 1 + \mu^2 P_3(\mu^2),$$

an expression which does not vanish identically.

Coefficient of α^2.

(150)
$$\begin{aligned} \rho_{200}'' + \theta_2\rho_{200} + \theta_3\sigma_{200} &= R_{200}, \\ \sigma_{200}'' + \theta_4\sigma_{200} + \theta_3\rho_{200} &= S_{200}. \end{aligned}$$

The right members of these equations have the following expressions:

$$\begin{aligned} R_{200} = {} & A_{100}^{(1)2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2] \\ S_{200} = {} & A_{100}^{(1)2}[\bar\theta_{200}\alpha_2^2 + \bar\theta_{110}\alpha_2\gamma_1 + \bar\theta_{020}\gamma_1^2] \\ & + A_{100}^{(1)}A_{100}^{(2)}[2\bar\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar\theta_{110}\{\alpha_2(\tau\gamma_4 + \gamma_3) + \gamma_1(\tau\alpha_3 + \alpha_4)\} \\ & \qquad + 2\bar\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + A_{100}^{(2)2}[\bar\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar\theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar\theta_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The initial conditions are

$$\rho_{200} = 0, \qquad \sigma_{200} = 0, \qquad \rho_{200}' = 0, \qquad \sigma_{200}' = 0.$$

Since equations (150) are linear, and their left members have the same form as (51), the solutions will have the form

$$
(151)\quad
\begin{aligned}
\rho_{200} &= A\alpha_2 + B\alpha_1 + C\alpha_3 + D[\tau\alpha_3 + \alpha_4] \\
&\qquad + A_{100}^{(1)^2}\phi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\phi_2(\tau) + A_{100}^{(2)^2}\phi_3(\tau), \\
\sigma_{200} &= A\gamma_1 + B\gamma_2 + C\gamma_4 + D[\tau\gamma_4 + \gamma_3] \\
&\qquad + A_{100}^{(1)^2}\psi_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}\psi_2(\tau) + A_{100}^{(2)^2}\psi_3(\tau).
\end{aligned}
$$

Imposing the initial conditions we find

$B = C = 0$,

$$
\begin{aligned}
A = {} & \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)^2} \\
&+ \frac{\psi_2'(0)\alpha_4(0) - \phi_2(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
&+ \frac{\psi_3'(0)\alpha_4(0) - \phi_3(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta} A_{100}^{(2)^2},
\end{aligned}
$$

$$
\begin{aligned}
D = {} & \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)^2} + \frac{\phi_2(0)\gamma_1'(0) - \psi_2'(0)\alpha_2(0)}{\Delta} A_{100}^{(1)}A_{100}^{(2)} \\
&+ \frac{\phi_3(0)\gamma_1'(0) - \psi_3'(0)\alpha_2(0)}{\Delta} A_{100}^{(2)^2}.
\end{aligned}
$$

Substituting these values in (151) we have

$$
(152)\quad
\begin{aligned}
\rho_{200} &= A_{100}^{(1)^2}x_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}x_2(\tau) + A_{100}^{(2)^2}x_3(\tau), \\
\sigma_{200} &= A_{100}^{(1)^2}y_1(\tau) + A_{100}^{(1)}A_{100}^{(2)}y_2(\tau) + A_{100}^{(2)^2}y_3(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(\tau) = \phi_1(\tau) &+ \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4'(0) + \gamma_3'(0)]}{\Delta}\alpha_2(\tau) \\
&+ \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\alpha_4(\tau) + \tau\alpha_3(\tau)],
\end{aligned}
$$

$$
\begin{aligned}
y_1(\tau) = \psi_1(\tau) &+ \frac{\psi_1'(0)\alpha_4(0) - \phi_1(0)[\gamma_4(0) + \gamma_3'(0)]}{\Delta}\gamma_1(\tau) \\
&+ \frac{\phi_1(0)\gamma_1'(0) - \psi_1'(0)\alpha_2(0)}{\Delta}[\gamma_3(\tau) + \tau\gamma_4(\tau)],
\end{aligned}
$$

and similar expressions for x_2, y_2 and x_3, y_3, which we shall find later [equation (172)] do not interest us. The characters of x_1 and y_1 are known with the exception of ϕ_1 and ψ_1 which we will now investigate. The functions ϕ_1 and ψ_1

are those portions of the solution of the differential equations (150) which depend upon the coefficients of $A_{100}^{(1)2}$. These coefficients are homogeneous of the second degree in $\alpha_2(\tau)$ and $\gamma_1(\tau)$. In R_{200} and S_{200} the expressions

θ_{200}, $\bar{\theta}_{110}$ and θ_{020} contain only cosines of even multiples of τ;

$\bar{\theta}_{200}$, θ_{110} and $\bar{\theta}_{020}$ contain only sines of odd multiples of τ;

$\alpha_2(\tau)$ has the form $\alpha_2 = \sum_k a_k \cos\,[(2k+1) \pm \lambda]\,\tau$;

$\gamma_1(\tau)$ has the form $\gamma_1 = \sum_k b_k \sin\,[2k \pm \lambda]\,\tau$.

Consequently, so far as the coefficients of $A_{100}^{(1)2}$ are concerned, R_{200} and S_{200} have the form

$$R_{200} = \{ \sum_k a_k^{(1)} \cos 2kt + \sum_k a_k^{(2)} \cos\,[2k \pm 2\lambda]\,\tau \}\, A_{100}^{(1)2} + \cdots,$$

$$S_{200} = \{ \sum_k b_k^{(1)} \sin (2k+1)\tau + \sum_k b_k^{(2)} \sin\,[(2k+1) \pm 2\lambda]\,\tau \}\, A_{100}^{(1)2} + \cdots.$$

By Theorem II terms involving 2λ give rise only to periodic terms in the solution whose period is $2K\pi$. By Theorem V those parts of ρ_{200} and σ_{200} depending upon the terms in R_{200} and S_{200} which are independent of λ have the form

$$\rho = p_1(\tau) + c\tau\alpha_3(\tau), \qquad \sigma = p_2(\tau) + c\tau\gamma_4(\tau)$$

respectively, where p_1 and p_2 are periodic with the period 2π. Consequently the characters of $x_1(\tau)$ and $y_1(\tau)$ are

$$(153) \qquad x_1(\tau) = \wp_1(\tau) + c_1\tau\alpha_3(\tau), \qquad y_1(\tau) = \wp_2(\tau) + c_1\tau\gamma_4(\tau),$$

where $\wp_1(\tau)$ and $\wp_2(\tau)$ are periodic with the period $2K\pi$.

Coefficient of $\alpha\delta$.

$$(154) \qquad \rho''_{110} + \theta_2\rho_{110} + \theta_3\sigma_{110} = R_{110}, \qquad \sigma''_{110} + \theta_4\sigma_{110} + \theta_3\rho_{110} = S_{110},$$

where

$$\begin{aligned} R_{110} = {} & 2A_{100}^{(1)}A_{010}^{(1)}\,[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}]\,[2\theta_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \theta_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ & \qquad + 2\theta_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + 2A_{100}^{(2)}A_{010}^{(2)}\,[\theta_{200}(\tau\alpha_3 + \alpha_4)^2 + \theta_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \theta_{020}(\tau\gamma_4 + \gamma_3)^2], \end{aligned}$$

$$\begin{aligned} S_{110} = {} & 2A_{100}^{(1)}A_{010}^{(1)}\,[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] \\ & + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}]\,[2\bar{\theta}_{200}\alpha_2(\tau\alpha_3 + \alpha_4) + \bar{\theta}_{110}\{\gamma_1(\tau\alpha_3 + \alpha_4) + \alpha_2(\tau\gamma_4 + \gamma_3)\} \\ & \qquad + 2\bar{\theta}_{020}\gamma_1(\tau\gamma_4 + \gamma_3)] \\ & + 2A_{100}^{(2)}A_{010}^{(2)}\,[\bar{\theta}_{200}(\tau\alpha_3 + \alpha_4)^2 + \bar{\theta}_{110}(\tau\alpha_3 + \alpha_4)(\tau\gamma_4 + \gamma_3) + \bar{\theta}_{020}(\tau\gamma_4 + \gamma_3)^2]. \end{aligned}$$

The functions R_{110} and S_{110} differ from R_{200} and S_{200} only in the constants A_{ijk}. The initial conditions impose the same conditional equations. Consequently the solutions differ only in the constants A_{ijk}, so that we can express them at once without computation:

$$(155)\quad \begin{aligned} \rho_{110} &= 2A_{100}^{(1)}A_{010}^{(1)}x_1(\tau) + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}]x_2(\tau) + 2A_{100}^{(2)}A_{010}^{(2)}x_3(\tau), \\ \sigma_{110} &= 2A_{100}^{(1)}A_{010}^{(1)}y_1(\tau) + [A_{100}^{(1)}A_{010}^{(2)} + A_{100}^{(2)}A_{010}^{(1)}]y_2(\tau) + 2A_{100}^{(2)}A_{010}^{(2)}y_3(\tau), \end{aligned}$$

where the $x_i(\tau)$ and $y_i(\tau)$ are the same functions of τ as before.

Coefficient of δ^2.

By symmetry with the coefficient of α^2 it is seen that

$$(156)\quad \begin{aligned} \rho_{020} &= A_{010}^{(1)^2}x_1(\tau) + A_{010}^{(1)}A_{010}^{(2)}x_2(\tau) + A_{010}^{(2)^2}x_3(\tau), \\ \sigma_{020} &= A_{010}^{(1)^2}y_1(\tau) + A_{010}^{(1)}A_{010}^{(2)}y_2(\tau) + A_{010}^{(2)^2}y_3(\tau). \end{aligned}$$

Coefficient of ϵ^2.

Since the coefficients of the first powers of α and δ were homogeneous in the A_{ijk} of the first degree, the coefficients of α^2, $\alpha\delta$ and δ^2 are homogeneous in the A_{ijk} of the second degree. The coefficient of the first power of ϵ is not homogeneous in the A_{ijk}, hence the second power is not homogeneous. But if the functions $\alpha_6(\tau)$ and $\gamma_6(\tau)$ were zero the coefficient of the first power of ϵ would be homogeneous. By symmetry therefore, we can at once write down the terms involving the A_{ijk} to the second degree. To these must be added terms in the first degree and one term independent of the A_{ijk}.

The differential equations are

$$(157)\quad \begin{aligned} &\rho''_{002} + \theta_2\rho_{002} + \theta_3\sigma_{002} = R_{002}, \\ &\sigma''_{002} + \theta_4\sigma_{002} + \theta_3\rho_{002} = S_{002}; \\ &R_{002} = \theta_{200}\rho_{001}^2 + \theta_{110}\rho_{001}\sigma_{001} + \theta_{020}\sigma_{001}^2 + \theta_{101}\rho_{001}, \\ &S_{002} = \bar\theta_{200}\rho_{001}^2 + \bar\theta_{110}\rho_{001}\sigma_{001} + \bar\theta_{020}\sigma_{001}^2. \end{aligned}$$

The terms involved in these expressions are shown in the following table:

R_{002}

Term	$A_{001}^{(1)^2}$	$A_{001}^{(1)}A_{001}^{(2)}$	$A_{001}^{(2)^2}$	$A_{001}^{(1)}$	$A_{001}^{(2)}$	1	Multiplied by
ρ_{001}^2	a_2^2	$2a_2(\tau a_3 + a_4)$	$(\tau a_3 + a_4)^2$	$2a_2a_6$	$2a_6(\tau a_3 + a_4)$	a_6^2	θ_{200}
$\rho_{001}\sigma_{001}$	$a_2\gamma_1$	$\gamma_1(\tau a_3 + a_4)$ $+ a_2(\tau\gamma_4 + \gamma_3)$	$(\tau a_3 + a_4)(\tau\gamma_4 + \gamma_3)$	$\gamma_1 a_6$ $+ a_2\gamma_6$	$a_6(\tau\gamma_4 + \gamma_3)$ $+ \gamma_6(\tau a_3 + a_4)$	$a_6\gamma_6$	θ_{110}
σ_{001}^2	γ_1^2	$2\gamma_1(\tau\gamma_4 + \gamma_3)$	$(\tau\gamma_4 + \gamma_3)^2$	$2\gamma_1\gamma_6$	$2(\tau\gamma_4 + \gamma_3)$	γ_6^2	θ_{020}
ρ_{001}				a_2	$(\tau a_3 + a_4)$	a_6	θ_{101}

In order to obtain the S_{002} it is necessary in the above table only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$.

The solutions of equations (157) may be expressed in the form

$$(158)\quad \begin{aligned} \rho_{002} &= A_{001}^{(1)^2} x_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} x_2(\tau) + A_{001}^{(2)^2} x_3(\tau) + A_{001}^{(1)} x_4(\tau) + A_{001}^{(2)} x_5(\tau) + x_6(\tau), \\ \sigma_{002} &= A_{001}^{(1)^2} y_1(\tau) + A_{001}^{(1)} A_{001}^{(2)} y_2(\tau) + A_{001}^{(2)^2} y_3(\tau) + A_{001}^{(1)} y_4(\tau) + A_{001}^{(2)} y_5(\tau) + y_6(\tau). \end{aligned}$$

The coefficients of $A_{001}^{(1)}$ in the differential equations are homogeneous of the first degree in α_2 and γ_1, every term of which involves the first multiple of $\lambda\tau$. Hence the solutions for these terms by Theorem II involve non-periodic terms, and we can write

$$(159)\quad \begin{aligned} x_4(\tau) &= \wp_3(\tau) + c_2 \tau \alpha_1(\tau) + c_3 \tau \alpha_3(\tau), \\ y_4(\tau) &= \wp_4(\tau) + c_2 \tau \gamma_2(\tau) + c_3 \tau \gamma_4(\tau), \end{aligned}$$

where $\wp_3(\tau)$ and $\wp_4(\tau)$ are periodic with the period $2K\pi$. As is seen from the table, $x_6(\tau)$ and $y_6(\tau)$ do not involve the λ. They have therefore the form

$$(160)\quad \begin{aligned} x_6(\tau) &= \wp_5(\tau) + c_4 \tau \alpha_3(\tau), \\ y_6(\tau) &= \wp_6(\tau) + c_4 \tau \gamma_4(\tau). \end{aligned}$$

It will be verified in (172) that we do not need to know the character of x_5 and y_5.

Coefficient of $\alpha\epsilon$.

The differential equations are

$$(161)\quad \begin{aligned} \rho_{101}'' + \theta_2 \rho_{101} + \theta_3 \sigma_{101} &= R_{101}, \\ \sigma_{101}'' + \theta_4 \sigma_{101} + \theta_3 \rho_{101} &= S_{101}. \end{aligned}$$

$$R_{101} = \theta_{200}\left[2\rho_{100}\rho_{001}\right] + \theta_{110}\left[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}\right] + \theta_{020}\left[2\sigma_{100}\sigma_{001}\right] + \theta_{101}\rho_{100},$$

$$S_{101} = \bar{\theta}_{200}\left[2\rho_{100}\rho_{001}\right] + \bar{\theta}_{110}\left[\rho_{100}\sigma_{001} + \rho_{001}\sigma_{100}\right] + \bar{\theta}_{020}\left[2\sigma_{100}\sigma_{001}\right].$$

The following table shows the character of the terms entering into these expressions:

R_{101}

	$2A_{100}^{(1)}A_{001}^{(1)}$	$A_{100}^{(1)}A_{001}^{(2)}+A_{100}^{(2)}A_{001}^{(1)}$	$2A_{100}^{(2)}A_{001}^{(2)}$	$A_{100}^{(1)}$	$A_{100}^{(2)}$	Multiplied by
$\rho_{100}\rho_{001}$	α_2^2	$2\alpha_2(\tau\alpha_3+\alpha_4)$	$(\tau\alpha_3+\alpha_4)^2$	$2\alpha_2\alpha_6$	$2\alpha_6(\tau\alpha_3+\alpha_4)$	θ_{200}
$\rho_{100}\sigma_{001}$ $+\rho_{001}\sigma_{100}$	$\alpha_2\gamma_1$	$\gamma_1(\tau\alpha_3+\alpha_4)$ $+\alpha_2(\tau\gamma_4+\gamma_3)$	$(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)$	$\alpha_6\gamma_1+\alpha_2\gamma_6$	$\alpha_6(\tau\gamma_4+\gamma_3)$ $+\gamma_6(\tau\alpha_3+\alpha_4)$	θ_{110}
$\sigma_{100}\sigma_{001}$	γ_1^2	$2\gamma_1(\tau\gamma_4+\gamma_3)$	$(\tau\gamma_4+\gamma_3)^2$	$2\gamma_1\gamma_6$	$2\gamma_6(\tau\gamma_4+\gamma_3)$	θ_{020}
ρ_{100}				α_2	$(\tau\alpha_3+\alpha_4)$	θ_{101}

In order to obtain S_{101} it is necessary only to change the θ_{ijk} in the last column into $\bar{\theta}_{ijk}$. This table shows that R_{101} and S_{101} differ from R_{002} and S_{002} only in the constants A_{ijk}. Since the initial conditions impose the same conditional equations as for the coefficient of ϵ^2 we can at once write down the solution

$$(162)\qquad \begin{aligned} \rho_{101} &= 2A_{100}^{(1)}A_{001}^{(1)}x_1(\tau) + [A_{100}^{(1)}A_{001}^{(2)} + A_{100}^{(2)}A_{001}^{(1)}]x_2(\tau) \\ &\qquad + 2A_{100}^{(2)}A_{001}^{(2)}x_3(\tau) + A_{100}^{(1)}x_4(\tau) + A_{100}^{(2)}x_5(\tau), \\ \sigma_{101} &= 2A_{100}^{(1)}A_{001}^{(1)}y_1(\tau) + [A_{100}^{(1)}A_{001}^{(2)} + A_{100}^{(2)}A_{001}^{(1)}]y_2(\tau) \\ &\qquad + 2A_{100}^{(2)}A_{001}^{(2)}y_3(\tau) + A_{100}^{(1)}y_4(\tau) + A_{100}^{(2)}y_5(\tau). \end{aligned}$$

Coefficient of $\delta\epsilon$.

This coefficient can be obtained by symmetry from the coefficient of $\alpha\epsilon$ by permutation of the first and second subscripts of the A_{ijk}. Therefore

$$(163)\qquad \begin{aligned} \rho_{011} &= 2A_{010}^{(1)}A_{001}^{(1)}x_1(\tau) + [A_{010}^{(1)}A_{001}^{(2)} + A_{010}^{(2)}A_{001}^{(1)}]x_2(\tau) \\ &\qquad + 2A_{010}^{(2)}A_{001}^{(2)}x_3(\tau) + A_{010}^{(1)}x_4(\tau) + A_{010}^{(2)}x_5(\tau), \\ \sigma_{101} &= 2A_{010}^{(1)}A_{001}^{(1)}y_1(\tau) + [A_{010}^{(1)}A_{001}^{(2)} + A_{010}^{(2)}A_{001}^{(1)}]y_2(\tau) \\ &\qquad + 2A_{010}^{(2)}A_{001}^{(2)}y_3(\tau) + A_{010}^{(1)}y_4(\tau) + A_{010}^{(2)}y_5(\tau). \end{aligned}$$

This concludes the computation of all terms up to the second order inclusive in α, δ and ϵ. In order to establish the existence of the periodic solutions it is not necessary to carry the computation further.

§ 16. *Existence of periodic orbits reëntrant only after many revolutions.*

We have chosen the initial conditions so that at $\tau = 0$ the particle is crossing the ρ-axis orthogonally. It is obvious geometrically that if at any future time it again crosses the ρ-axis perpendicularly the orbit will be a closed one and the motion in it will be periodic. The conditions that the particle shall cross the ρ-axis perpendicularly at $\tau = T$ is that at this epoch

$$\rho' = \sigma = 0.$$

The equations of variation have the period $2K\pi$. Therefore we shall choose $T = K\pi$. Since ρ is an even series in τ, and σ is an odd series, all the purely periodic terms in ρ' and σ are sines and consequently vanish at $\tau = K\pi$. The terms which do not vanish must carry τ as a factor. The conditions for periodicity give us two equations, namely, at $\tau = K\pi$,

$$(164)\qquad \begin{aligned} \rho' = 0 &= a_{100}\alpha + a_{010}\delta + a_{001}\epsilon + a_{200}\alpha^2 + a_{110}\alpha\delta \\ &\qquad + a_{020}\delta^2 + a_{101}\alpha\epsilon + a_{011}\delta\epsilon + a_{002}\epsilon^2 + \cdots, \\ \sigma = 0 &= b_{100}\alpha + b_{010}\delta + b_{001}\epsilon + b_{200}\alpha^2 + b_{110}\alpha\delta \\ &\qquad + b_{020}\delta^2 + b_{101}\alpha\epsilon + b_{011}\delta\epsilon + b_{002}\epsilon^2 + \cdots, \end{aligned}$$

where a_{ijk} and b_{ijk} are the values derived from the series just computed. Their values are as follows:

$$
(165) \qquad \begin{aligned}
a_{100} &= A^{(2)}_{100} u, & a_{010} &= A^{(2)}_{010} u, & a_{001} &= A^{(2)}_{001} u,\\
b_{100} &= A^{(2)}_{100} v, & b_{010} &= A^{(2)}_{010} v, & b_{001} &= A^{(2)}_{001} v,
\end{aligned}
$$

$$
\begin{aligned}
a_{200} &= A^{(1)^2}_{100}\bar{x}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{x}_2 + A^{(2)^2}_{100}\bar{x}_3,\\
b_{200} &= A^{(1)^2}_{100}\bar{y}_1 + A^{(1)}_{100}A^{(2)}_{100}\bar{y}_2 + A^{(2)^2}_{100}\bar{y}_3,
\end{aligned}
$$

$$
\begin{aligned}
a_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{x}_3,\\
b_{110} &= 2A^{(1)}_{100}A^{(1)}_{010}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{010} + A^{(2)}_{100}A^{(1)}_{010}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{010}\bar{y}_3,
\end{aligned}
$$

$$
\begin{aligned}
a_{020} &= A^{(1)^2}_{010}\bar{x}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{x}_2 + A^{(2)^2}_{010}\bar{x}_3,\\
b_{020} &= A^{(1)^2}_{010}\bar{y}_1 + A^{(1)}_{010}A^{(2)}_{010}\bar{y}_2 + A^{(\ )^2}_{010}\bar{y}_3,
\end{aligned}
$$

$$
\begin{aligned}
a_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{100}\bar{x}_4 + A^{(2)}_{100}\bar{x}_5,\\
b_{101} &= 2A^{(1)}_{100}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{100}A^{(2)}_{001} + A^{(2)}_{100}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{100}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{100}\bar{y}_4 + A^{(2)}_{100}\bar{y}_5,
\end{aligned}
$$

$$
\begin{aligned}
a_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{x}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{x}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{x}_3 + A^{(1)}_{010}\bar{x}_4 + A^{(2)}_{010}\bar{x}_5,\\
b_{011} &= 2A^{(1)}_{010}A^{(1)}_{001}\bar{y}_1 + [A^{(1)}_{010}A^{(2)}_{001} + A^{(2)}_{010}A^{(1)}_{001}]\bar{y}_2 + 2A^{(2)}_{010}A^{(2)}_{001}\bar{y}_3 + A^{(1)}_{010}\bar{y}_4 + A^{(2)}_{010}\bar{y}_5,
\end{aligned}
$$

$$
\begin{aligned}
a_{002} &= A^{(1)^2}_{001}\bar{x}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{x}_2 + A^{(2)^2}_{001}\bar{x}_3 + A^{(1)}_{001}\bar{x}_4 + A^{(2)}_{001}\bar{x}_5 + \bar{x}_6,\\
b_{002} &= A^{(1)^2}_{001}\bar{y}_1 + A^{(1)}_{001}A^{(2)}_{001}\bar{y}_2 + A^{(2)^2}_{001}\bar{y}_3 + A^{(1)}_{001}\bar{y}_4 + A^{(2)}_{001}\bar{y}_5 + \bar{y}_6,
\end{aligned}
$$

where

$$
(166) \qquad \begin{aligned}
u &= K\pi\frac{d\alpha_3}{d\tau}, & v &= k\pi\gamma_4,\\
\bar{x}_i &= \frac{dx_i}{d\tau}, & \bar{y}_i &= y_i, \qquad \text{at } \tau = K\pi.
\end{aligned}
$$

Let us solve now the equation (164), $\rho' = 0$, for ϵ as a power series in α and δ. We obtain

$$
(167) \qquad \epsilon = \epsilon_{10}\alpha + \epsilon_{01}\delta + \epsilon_{20}\alpha^2 + \epsilon_{11}\alpha\delta + \epsilon_{02}\delta^2 + \cdots,
$$

where the coefficients ϵ_{ij} have the following values:

$$
\begin{aligned}
\epsilon_{10} &= -\frac{a_{100}}{a_{001}},\\
\epsilon_{01} &= -\frac{a_{010}}{a_{001}},\\
\epsilon_{20} &= -\frac{a_{200}a_{001}^2 - a_{100}a_{101}a_{001} + a_{002}a_{100}^2}{a_{001}^3},
\end{aligned}
$$

$$\epsilon_{11} = -\frac{a_{110}a_{001}^2 - a_{011}a_{100}a_{001} - a_{101}a_{010}a_{001} + 2a_{002}a_{100}a_{010}}{a_{001}^3},$$

$$\epsilon_{02} = -\frac{a_{020}a_{001}^2 - a_{011}a_{010}a_{001} + a_{002}a_{010}^2}{a_{001}^3}.$$

Solving now equation (164), $\sigma = 0$, for ϵ in terms of α and δ, we obtain

$$\epsilon = \bar{\epsilon}_{10}\alpha + \bar{\epsilon}_{01}\delta + \bar{\epsilon}_{20}\alpha^2 + \bar{\epsilon}_{11}\alpha\delta + \bar{\epsilon}_{02}\delta^2 + \cdots, \tag{168}$$

where the $\bar{\epsilon}_{ij}$ have the same expressions in the b_{klm} as the ϵ_{ij} have in the a_{klm}. Subtracting (167) from (168), we have

$$0 = [\bar{\epsilon}_{10} - \epsilon_{10}]\alpha + [\bar{\epsilon}_{01} - \epsilon_{01}]\delta + [\bar{\epsilon}_{20} - \epsilon_{20}]\alpha^2 + [\bar{\epsilon}_{11} - \epsilon_{11}]\alpha\delta + [\bar{\epsilon}_{02} - \epsilon_{02}]\delta^2 + \cdots. \tag{169}$$

We must now examine the coefficients of this series.

$$\begin{aligned} [\bar{\epsilon}_{10} - \epsilon_{10}] &= \frac{a_{100}}{a_{001}} - \frac{b_{100}}{b_{001}} = \frac{A_{100}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{100}^{(2)}v}{A_{001}^{(2)}v} = 0, \\ [\bar{\epsilon}_{01} - \epsilon_{10}] &= \frac{a_{010}}{a_{001}} - \frac{b_{010}}{b_{001}} = \frac{A_{010}^{(2)}u}{A_{001}^{(2)}u} - \frac{A_{010}^{(2)}v}{A_{001}^{(2)}v} = 0. \end{aligned} \tag{170}$$

Both of the linear terms therefore vanish. The computation of the second degree terms is somewhat more complicated. It simplifies matters somewhat if we observe that the $\bar{\epsilon}_{ij}$ have the same expressions in v and the $\bar{y}_i$ as the ϵ_{ij} have in u and $\bar{x}_i$. It is sufficient therefore to compute one and derive the other from it. From (165) we have

$$-\epsilon_{20} = \frac{a_{200}}{a_{001}} - \frac{a_{100}a_{101}}{a_{001}^2} + \frac{a_{002}a_{100}^2}{a_{001}^3}.$$

Substituting in these fractions the values of the a_{ijk} from (165) we get

$$\begin{aligned} \frac{a_{200}}{a_{001}} &= \frac{1}{A_{001}^{(2)^3}}\left[A_{100}^{(1)^2}A_{001}^{(2)^2}\frac{\bar{x}_1}{u} + A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)^2}\frac{\bar{x}_2}{u} + A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u}\right], \\ -\frac{a_{100}a_{101}}{a_{001}^2} &= \frac{1}{A_{001}^{(2)^3}}\Bigg[-2A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_1}{u} + [-A_{100}^{(2)^2}A_{001}^{(1)}A_{001}^{(2)} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)^2}]\frac{\bar{x}_2}{u} \\ &\qquad - 2A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u} - A_{100}^{(1)}A_{100}^{(2)}A_{001}^{(2)}\frac{\bar{x}_4}{u} - A_{100}^{(2)^2}A_{001}^{(2)}\frac{\bar{x}_5}{u}\Bigg], \\ \frac{a_{002}a_{100}^2}{a_{001}^3} &= \frac{1}{A_{001}^{(2)^3}}\Bigg[A_{100}^{(2)^2}A_{001}^{(1)^2}\frac{\bar{x}_1}{u} + A_{100}^{(2)^2}A_{001}^{(1)}A_{001}^{(2)}\frac{\bar{x}_2}{u} + A_{100}^{(2)^2}A_{001}^{(2)^2}\frac{\bar{x}_3}{u} \\ &\qquad + A_{100}^{(2)^2}A_{001}^{(1)}\frac{\bar{x}_4}{u} + A_{100}^{(2)^2}A_{001}^{(2)}\frac{\bar{x}_5}{u}A_{100}^{(2)^2}\frac{\bar{x}_6}{u}\Bigg]. \end{aligned} \tag{171}$$

For the sum of these three expressions, there results

$$
(172)\qquad -\epsilon_{20} = \frac{1}{A^{(2)^3}_{001}}\Bigg[\,[A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}]^2\frac{\bar{x}_1}{u} - [A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}]\,A^{(2)}_{100}\frac{\bar{x}_4}{u} + A^{(2)^2}_{100}\frac{\bar{x}_6}{u}\Bigg],
$$

the coefficients of $\bar{x}_2/u$, $\bar{x}_3/u$ and $\bar{x}_5/u$ vanishing identically. This is the reason why it was not necessary to compute x_2, x_3, and x_5 in (152) and (158).

Changing the $\bar{x}_i$ into $\bar{y}_i$ and u into v gives us $-\bar{\epsilon}_{20}$. Hence

$$
(173)\qquad [\bar{\epsilon}_{20} - \epsilon_{20}] = \frac{1}{A^{(2)^3}_{001}}\Bigg\{[A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}]^2\left[\frac{\bar{x}_1}{u} - \frac{\bar{y}_1}{v}\right] - A^{(2)}_{100}[A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}]\left[\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v}\right] + A^{(2)^2}_{100}\left[\frac{\bar{x}_6}{u} - \frac{\bar{y}_6}{v}\right]\Bigg\}.
$$

But $[\bar{x}_1/u - \bar{y}_1/v]$ and $[\bar{x}_6/u - \bar{y}_6/v]$ both vanish since

$$
\bar{x}_1 = c_1 u, \qquad \bar{y}_1 = c_1 v, \qquad \bar{x}_6 = c_4 u, \qquad \bar{y}_6 = c_4 v,
$$

as is readily seen from (153) and (160). It is also seen on referring to (159) that $[\bar{x}_4/u - \bar{y}_4/v]$ does not vanish, but is equal to

$$
\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v} = c_2\left[\frac{1}{u}\frac{d\alpha_1}{d\tau} - \frac{\gamma_2}{v}\right]_{\tau=K\pi}.
$$

Hence

$$
(174)\qquad [\bar{\epsilon}_{20} - \epsilon_{20}] = -\frac{A^{(2)}_{100}[A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}]}{A^{(2)^3}_{001}}\left[\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v}\right].
$$

Without repeating the details of the computation we find in a similar way

$$
(175)\qquad [\bar{\epsilon}_{11} - \epsilon_{11}] = -\frac{A^{(2)}_{010}[A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001}] + A^{(2)}_{100}[A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}]}{A^{(2)^3}_{001}}\left[\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v}\right],
$$

$$
(176)\qquad [\bar{\epsilon}_{02} - \epsilon_{02}] = -\frac{A^{(2)}_{010}[A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}]}{A^{(2)^3}_{001}}\left[\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v}\right].
$$

Substituting these values in (169), we find that the second degree terms in α and δ are factorable. Thus (169) becomes

$$
(177)\qquad 0 = \frac{1}{A^{(2)^3}_{001}}\left[\frac{\bar{x}_4}{u} - \frac{\bar{y}_4}{v}\right][A^{(2)}_{100}\alpha + A^{(2)}_{010}\delta + \cdots][(A^{(1)}_{1\ 0}A^{(2)}_{001} - A^{(2)}_{100}A^{(1)}_{001})\alpha + (A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001})\delta + \cdots].
$$

There are therefore two real solutions for δ as power series in α. Substituting these values of δ in (168) we find the two corresponding values of ϵ. In this manner we find:

First solution.

$$(178)\qquad \begin{aligned} \delta &= -\frac{A^{(1)}_{100}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}{A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}\alpha + \cdots = \frac{\gamma_5' - a\gamma_3'}{\alpha_5 - a\alpha_4}\alpha + \cdots = \frac{\gamma_6'(0)}{\alpha_6(0)}\alpha + \cdots, \\ \epsilon &= \frac{A^{(1)}_{100}A^{(2)}_{010} - A^{(2)}_{100}A^{(1)}_{010}}{A^{(1)}_{010}A^{(2)}_{001} - A^{(2)}_{010}A^{(1)}_{001}}\alpha + \cdots = \frac{1}{\alpha_5 - a\alpha_4}\alpha + \cdots = \frac{1}{\alpha_6(0)}\alpha + \cdots. \end{aligned}$$

Second solution.

$$(179)\qquad \begin{aligned} \delta &= -\frac{A^{(2)}_{100}}{A^{(2)}_{010}}\alpha + \cdots = \frac{\gamma_1'(0)}{\alpha_2(0)}\alpha + \cdots = \gamma_2'(0)\cdot\alpha + \cdots, \\ \epsilon &= -\frac{A^{(2)}_{100}}{A^{(2)}_{001}}\alpha - \frac{A^{(2)}_{010}}{A^{(2)}_{001}}\delta + \cdots = 0\cdot\alpha + \cdots, \end{aligned}$$

where α_6 and γ_6 are the quantities defined in (147), and $\gamma_6'(0)$ is the value of $d\gamma_6/d\tau$ for $\tau = 0$. Thus one solution for ϵ begins with the first power of α while the other certainly does not begin before the second, but in both solutions δ begins with the first power of α.

§ 17. *Construction of the solutions having the period $2K\pi$.*

We have just proved the existence of series for ρ, σ and ϵ proceeding in powers of the initial value of ρ (let us call this value e).* The series for ρ and σ are periodic in τ with the period $2K\pi$, and since this condition holds for all values of e sufficiently small each coefficient is separately periodic. The series for ρ is even in τ, and the series for σ is odd in τ. These series have the form

$$(180)\qquad \begin{aligned} \rho &= \rho_1 e + \rho_2 e^2 + \rho_3 e^3 + \cdots, \\ \sigma &= \sigma_1 e + \sigma_2 e^2 + \sigma_3 e^3 + \cdots, \\ \epsilon &= \epsilon_1 e + \epsilon_2 e^2 + \epsilon_3 e^3 + \cdots. \end{aligned}$$

We may substitute these series in the differential equations and integrate the coefficients of each power of e step by step, and determine the constants in such a way that ρ and σ shall be periodic, and satisfy the initial conditions

$$(181)\qquad \rho(0) = e, \qquad \sigma(0) = 0, \qquad \rho'(0) = 0, \qquad \sigma'(0) = \delta,$$

where δ is a constant at present unknown but to be determined in the process.

Substituting the series (180) in the differential equations (138), we find for the coefficient of the first power of e

$$(182)\qquad \rho_1'' + \theta_2\rho_1 + \theta_3\sigma_1 = \theta_{001}\epsilon_1, \qquad \sigma_1'' + \theta_4\sigma_1 + \theta_3\rho_1 = 0.$$

* The reason for changing α to e is that this parameter corresponds to the eccentricity in the two-body problem.

Since by the condition of orthogonality ρ must be even in τ and σ odd in τ, the solution complying with the condition is

$$(183)\quad \begin{aligned} \rho_1 &= A^{(1)}\alpha_2(\tau) + D^{(1)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_1[a\tau\alpha_3(\tau) + \alpha_5(\tau)], \\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + D^{(1)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_1[a\tau\gamma_4(\tau) + \gamma_5(\tau)], \end{aligned}$$

where $\alpha_5(\tau)$ contains only cosines of even multiples of τ and $\gamma_5(\tau)$ contains only sines of odd multiples of τ.

In order that this solution shall be periodic it is necessary and sufficient that

$$D^{(1)} = -a\epsilon_1.$$

Upon imposing this condition the solution (183) becomes [see eq. (146)]

$$(184)\quad \begin{aligned} \rho_1 &= A^{(1)}\alpha_2(\tau) + \epsilon_1[\alpha_5(\tau) - a\alpha_4(\tau)] = A^{(1)}\alpha_2(\tau) + \epsilon_1\alpha_6(\tau), \\ \sigma_1 &= A^{(1)}\gamma_1(\tau) + \epsilon_1[\gamma_5(\tau) - a\gamma_3(\tau)] = A^{(1)}\gamma_1(\tau) + \epsilon_1\gamma_6(\tau). \end{aligned}$$

It remains to impose the initial condition that $\rho_1 = 1$ at $\tau = 0$. From this condition we get

$$(185)\qquad 1 = A^{(1)}\bar{\alpha}_2 + \epsilon_1\bar{\alpha}_6,$$

where $\bar{\alpha}_2$ and $\bar{\alpha}_6$ denote the values of these functions when $\tau = 0$.

Coefficient of e^2.

$$(186)\quad \begin{aligned} \rho_2'' + \theta_2\rho_2 + \theta_3\sigma_2 &= \theta_{001}\epsilon_2 + \theta_{101}\epsilon_1\rho_1 + \theta_{200}\rho_1^2 + \theta_{110}\rho_1\sigma_1 + \theta_{020}\sigma_1^2 = R_2, \\ \sigma_2'' + \theta_4\sigma_2 + \theta_3\rho_2 &= \qquad\qquad\qquad\quad \bar{\theta}_{200}\rho_1^2 + \bar{\theta}_{110}\rho_1\sigma_1 + \bar{\theta}_{020}\sigma_1^2 = S_2. \end{aligned}$$

Every term of R_2 and S_2 contains either $A^{(1)}$, ϵ_1 or ϵ_2 as a factor. Arranged in this manner we have the expansions

$$\begin{aligned} R_2 = {} & A^{(1)^2}[\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] \\ & + A^{(1)}\epsilon_2[2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\theta_{020}\gamma_1\gamma_6 + \theta_{101}\alpha_2] \\ & + \epsilon_1^2[\theta_{200}\alpha_6^2 + \theta_{110}\alpha_6\gamma_6 + \theta_{020}\gamma_6^2 + \theta_{101}\alpha_6] + \epsilon_2[\theta_{001}], \\ S_2 = {} & A^{(1)^2}[\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{110}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2] + A^{(1)}\epsilon_1[2\bar{\theta}_{200}\alpha_2\alpha_6 + \bar{\theta}_{110}(\gamma_1\alpha_6 + \alpha_2\gamma_6) + 2\bar{\theta}_{020}\gamma_1\gamma_6] \\ & + \epsilon_1^2[\bar{\theta}_{200}\alpha_6^2 + \bar{\theta}_{110}\alpha_6\gamma_6 + \bar{\theta}_{020}\gamma_6^2]. \end{aligned}$$

In order to understand the character of the solution of these equations we must examine the character of the various terms. The coefficient of $A^{(1)^2}$ in both R_2 and S_2 is homogeneous of the second degree in α_2 and γ_1. Its expansion therefore involves terms carrying $2\lambda\tau$ and terms independent of λ. By Theorem II the solution for the terms in $2\lambda\tau$ is periodic. The terms independent of λ are cosines of even multiples of τ in R_2 and sines of odd multiples of τ in

S_2. These terms have the same character as those in the coefficients of ϵ_1^2 and ϵ_2 and may be considered under the discussion of those terms.

The coefficients of $A^{(1)}\epsilon_1$ in both R_2 and S_2 are homogeneous of the first degree in α_2 and γ_1, all terms of which carry the first multiple of $\lambda\tau$. By Theorem VII the solution for ρ_2 will carry the term $\tau\alpha_1(\tau)$, and for σ_2 the term $\tau\gamma_2(\tau)$. Non-periodic terms of this character do not arise elsewhere in the solution, hence, in order to avoid them, we must take either $A^{(1)} = 0$ or $\epsilon_1 = 0$. If we choose $A^{(1)} = 0$, then, by (185), ϵ_1 is determined so that we must have $\epsilon_1 = 1/\bar{\alpha}_6$, thus agreeing with the first solution (178) of the existence proof. But if we choose $\epsilon_1 = 0$ so that by (185) $A^{(1)} = 1/\bar{\alpha}_2$ we are in agreement with the second solution (179) of the existence proof. We will commence by developing the first solution. This necessitates the choice $A^{(1)} = 0$, $\epsilon_1 = 1/\bar{\alpha}_6$.

First solution.

Since $A^{(1)} = 0$ all terms in R_2 and S_2 which carry $\lambda\tau$ or any multiple of it vanish. There remains

$$(187)\qquad \begin{aligned} R_2 &= \epsilon_1^2\left[\theta_{200}\alpha_6^2 + \theta_{110}\alpha_6\gamma_6 + \theta_{020}\gamma_6^2 + \theta_{101}\alpha_6\right] + \epsilon_2\theta_{001}, \\ S_2 &= \epsilon_1^2\left[\bar{\theta}_{200}\alpha_6^2 + \bar{\theta}_{110}\alpha_6\gamma_6 + \bar{\theta}_{020}\gamma_6^2\right]. \end{aligned}$$

We have also

$$(188)\qquad \rho_1 = \frac{\alpha_6(\tau)}{\bar{\alpha}_6}, \qquad \sigma_1 = \frac{\gamma_6(\tau)}{\bar{\alpha}_6}, \qquad \epsilon_1 = \frac{1}{\bar{\alpha}_6}.$$

It is easy to characterize R_2 and S_2. R_2 contains only cosines of even multiples of τ, and S_2 contains only sines of odd multiples of τ. Since ρ_2 is even in τ and σ_2 odd, the solution satisfying this condition is

$$\rho_2 = A^{(2)}\alpha_2(\tau) + D^{(2)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_2(\tau) + a_2\tau\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\alpha_3(\tau)],$$

$$\sigma_2 = A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)].$$

In this solution the terms are grouped according to their origin. The first two terms are the complementary function. The third arises from the terms carrying ϵ_1^2 as a factor. The fourth arises from the terms carrying ϵ_2 as a factor. a_2 is a constant depending upon the coefficients of ϵ_1^2 in the differential equations. $\alpha_5(\tau)$ and $\gamma_5(\tau)$ are the same functions as in the coefficient of the first power of e. By Theorems IV and V, $\eta_2(\tau)$ and $\zeta_2(\tau)$ are periodic functions of τ with the period 2π and so constituted that $\eta_2(\tau)$ contains only cosines of even multiples of τ, and $\zeta_2(\tau)$ contains only sines of odd multiples.

In order that ρ_2 and σ_2 shall be periodic we must have

$$D^{(2)} = -a_2 - a\epsilon_2,$$

and this makes

$$\rho_2 = A^{(2)}\alpha_2(\tau) + \epsilon_2\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau),$$
$$\sigma_2 = A^{(2)}\gamma_1(\tau) + \epsilon_2\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau).$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$, and this determines ϵ_2 so that

$$\epsilon_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2 - A^{(2)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

$A^{(2)}$ is not yet determined, but it is obvious that it must be zero in order to satisfy the periodicity condition on the coefficient of e^3 in which the non-periodic parts arising from terms involving the first multiple of $\lambda\tau$ carry $A^{(2)}$ as a factor. Hence the only way to avoid non-periodic terms of this character is to choose $A^{(2)} = 0$. Anticipating this step then we have

$$\rho_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\alpha_6(\tau) + \eta_2(\tau) - a_2\alpha_4(\tau), \tag{191}$$
$$\sigma_2 = \frac{a_2\bar{\alpha}_4 - \bar{\eta}_2}{\bar{\alpha}_6}\gamma_6(\tau) + \zeta_2(\tau) - a_2\gamma_3(\tau).$$

Hence ρ_2 contains only cosines of even multiples of τ, and σ_2 contains only sines of odd multiples of τ.

It remains only to show that this process of integration can be carried on indefinitely. Assuming that up to and including ρ_{i-1} and σ_{i-1} every ρ_j and σ_j is periodic with the period 2π and that the ρ_j contain only cosines of even multiples of τ and σ_j only sines of odd multiples of τ, except that ρ_{i-1} contains the term $A^{(i-1)}\alpha_2(\tau)$ and σ_{i-1} contains the term $A^{(i-1)}\gamma_1(\tau)$, it will be shown that the same conditions obtain for the next succeeding step. For ρ_i and σ_i we have from the differential equations (138)

$$\begin{aligned}\rho_i'' + \theta_2\rho_i + \theta_3\sigma_i = \theta_{001}\epsilon_i + A^{(i-1)}[\theta_{101}\epsilon_1\alpha_2 + 2\theta_{200}\rho_1\alpha_2 \\ + \theta_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\theta_{030}\sigma_1\gamma_1] + \Phi_i,\end{aligned} \tag{192}$$
$$\sigma_i'' + \theta_4\sigma_i + \theta_3\rho_i = A^{(i-1)}[2\bar{\theta}_{200}\rho_1\alpha_2 + \bar{\theta}_{110}(\rho_1\gamma_1 + \sigma_1\alpha_2) + 2\bar{\theta}_{020}\sigma_1\gamma_1] + \Psi_i.$$

From the properties of the differential equations it is readily seen that Φ_i contains only known terms all of which are cosines of even multiples of τ, and that Ψ_i contains only known terms all of which are sines of odd multiples of τ. The coefficients of $A^{(i-1)}$ are homogeneous of the first degree in α_2 and γ_1, and consequently each term involves a first multiple of $\lambda\tau$. Their solution gives rise to non-periodic terms of the form $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. They carry $A^{(i-1)}$ as a factor, and since terms of this type arise nowhere else in the solution we can make them disappear only by putting $A^{(i-1)} = 0$. The solution for (192) then has the form

$$(193)\quad \begin{aligned} \rho_i &= A^{(i)}\alpha_2(\tau) + D^{(i)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_i(\tau) + a_i\tau\alpha_3(\tau)] + \epsilon_i[\alpha_5(\tau) + a\tau\alpha_3(\tau)], \\ \sigma_i &= A^{(i)}\gamma_1(\tau) + D^{(i)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_i(\tau) + a_i\tau\gamma_4(\tau)] + \epsilon_i[\gamma_5(\tau) + a\tau\gamma_4(\tau)], \end{aligned}$$

where $\eta_i(\tau)$ and $\zeta_i(\tau)$ are periodic with the period 2π, and by Theorem V, $\eta_i(\tau)$ contains only cosines of even multiples of τ, and $\zeta_i(\tau)$ contains only sines of odd multiples of τ.

For ρ_i and σ_i to be periodic it is necessary and sufficient that

$$D^{(i)} = -a_i - a\epsilon_i,$$

which makes

$$(194)\quad \begin{aligned} \rho_i &= A^{(i)}\alpha_2(\tau) + \eta_i(\tau) - a_i\alpha_4(\tau) + \epsilon_i\alpha_6(\tau), \\ \sigma_i &= A^{(i)}\gamma_1(\tau) + \zeta_i(\tau) - a_i\gamma_3(\tau) + \epsilon_i\gamma_6(\tau). \end{aligned}$$

From the initial conditions we must have $\rho_i(0) = 0$. This condition determines ϵ_i to be

$$\epsilon_i = \frac{a_i\bar{\alpha}_4 - \bar{\eta}_i - A^{(i)}\bar{\alpha}_2}{\bar{\alpha}_6}.$$

Thus the constants are uniquely determined. The ρ_i and σ_i have the properties assumed for those having lower subscripts, and the process of integration can be continued indefinitely. Every $A^{(j)}$ is zero. Since no terms involving the $\lambda\tau$ enter, the solution has the period 2π. But the orbits represented belong to the class of generating orbits from which we started. In other words, we set out with a generating orbit for which the initial distance was, let us say, r_0, and we have found another generating orbit for which the initial distance is $r_0 + e$ (e arbitrary). There is nothing surprising in this, for r_0 is a function of an arbitrary constant β. Let us suppose we had started with a definite value of β, let us say β_0. This gives us a definite generating orbit with a definite initial distance r_0. Let us seek now the generating orbit for which the initial distance is $r_0 + e$. If e is sufficiently small we can evidently give an increment ϵ to β which will increase r_0 by the amount e. We have

$$r_0 = f(\beta_0),$$

$$r_0 + e = f(\beta_0 + \epsilon).$$

Expanding the right member of the last equation by Taylor's theorem we have

$$e = \frac{\partial f}{\partial \beta_0}\epsilon + \tfrac{1}{2}\frac{\partial^2 f}{\partial \beta_0^2}\epsilon^2 + \cdots,$$

and by inversion, since

$$\left.\frac{\partial f}{\partial \beta}\right|_{\beta=\beta_0} \neq 0,$$

$$\epsilon = \frac{1}{\dfrac{\partial f}{\partial \beta_0}} e + \cdots,$$

which we might write

$$\epsilon = e_1 e + e_2 e^2 + \cdots.$$

Then by substituting in the generating orbit

$$\beta = \beta_0 + e_1 e + e_2 e^2 + \cdots$$

and arranging as a power series in e, we would obtain the generating orbit which we sought. As these are the same conditions that are imposed when we seek new orbits through the equations of variation it is to be anticipated that one of the class of generating orbits satisfies the conditions.

Second solution.

We return now to equation (186) where the two methods of satisfying the periodicity condition presented themselves, and we shall continue with the second solution. We choose $\epsilon_1 = 0$, which we found determines $A^{(1)}$ to be $A^{(1)} = 1/\bar{\alpha}_2$. From (82) it is seen that $\bar{\alpha}_2 = \alpha_2(0) = 1$, and therefore $A^{(1)} = 1$. Hence in the second solution

(195) $$\rho_1 = \alpha_2(\tau), \qquad \sigma_1 = \gamma_1(\tau).$$

Using these values of $A^{(1)}$ and ϵ_1, R_2 and S_2 of (186) become

(196) $$\begin{aligned} R_2 &= [\theta_{200}\alpha_2^2 + \theta_{110}\alpha_2\gamma_1 + \theta_{020}\gamma_1^2] + \epsilon_2\theta_{001}, \\ S_2 &= [\bar{\theta}_{200}\alpha_2^2 + \bar{\theta}_{101}\alpha_2\gamma_1 + \bar{\theta}_{020}\gamma_1^2]. \end{aligned}$$

All of the terms in these expressions except $\epsilon_2\theta_{001}$ are of the second degree in α_2 and γ_1. Therefore they involve terms carrying $2\lambda\tau$ and terms independent of $\lambda\tau$. θ_{001} also is independent of $\lambda\tau$. In the solution the terms depending upon $2\lambda\tau$ are periodic by Theorem II. As for the terms independent of λ, R_2 contains only cosines of even multiples of τ and S_2 contains only sines of odd multiples of τ. These terms give rise to non-periodic terms in the solution which has the form

(197) $$\begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + D^{(2)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \phi_2(\lambda, \tau) \\ &\qquad + [\eta_2(\tau) + a_2\tau\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\alpha_3(\tau)]; \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \psi_2(\lambda, \tau) \\ &\qquad + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)]. \end{aligned}$$

In these equations $\phi_2(\lambda, \tau)$ and $\psi_2(\lambda, \tau)$ are the periodic terms involving λ, η_2 and ζ_2 are the periodic terms with the period 2π; a_2 the constant belonging to the non-periodic part, and the coefficients of ϵ_2 are the solutions arising from the coefficient of ϵ_2 in the differential equations. In order that this solution (197) shall be periodic it is necessary that

$$D^{(2)} = -a_2 - a\epsilon_2,$$

and this reduces ρ_2 and σ_2 to

$$(198)\qquad \begin{aligned} \rho_2 &= A^{(2)}\alpha_2(\tau) + \phi_2(\lambda, \tau) + \eta_2(\tau) - a_2\alpha_4(\tau) + \epsilon_2\alpha_6(\tau), \\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + \psi_2(\lambda, \tau) + \zeta_2(\tau) - a_2\gamma_3(\tau) + \epsilon_2\gamma_6(\tau). \end{aligned}$$

In order that we may satisfy the initial conditions we must have $\rho_2(0) = 0$. Hence, since $\alpha_2(0) = 1$,

$$A^{(2)} = -\bar{\phi}_2 - \bar{\eta}_2 + a_2\bar{\alpha}_4 - \epsilon_2\bar{\alpha}_6.$$

The constant ϵ_2 is determined by the periodicity condition for the coefficient of e^3.

Coefficient of e^3.

$$(199)\qquad \rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = R_3, \qquad \sigma_3'' + \theta_4\sigma_3 + \theta_3\rho_3 = S_3,$$

where

$$\begin{aligned} R_3 = {} & \theta_{001}\epsilon_3 + \theta_{101}\epsilon_2\rho_1 + 2\theta_{200}\rho_1\rho_2 + \theta_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\theta_{020}\sigma_1\sigma_2 + \theta_{300}\rho_1^3 \\ & + \theta_{210}\rho_1^2\sigma_1 + \theta_{120}\rho_1\sigma_1^2 + \theta_{030}\sigma_1^3, \\ S_3 = {} & 2\bar{\theta}_{200}\rho_1\rho_2 + \bar{\theta}_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\bar{\theta}_{020}\sigma_1\sigma_2 + \bar{\theta}_{300}\rho_1^3 \\ & + \bar{\theta}_{210}\rho_1^2\sigma_1 + \bar{\theta}_{120}\rho_1\sigma_1^2 + \bar{\theta}_{030}\sigma_1^3. \end{aligned}$$

In classifying the terms which belong to the expansion of R_3 and S_3 we bear in mind:

First. The θ_{ijk} in R_3 involve only cosines of even multiples of τ, except those which are coefficients of odd powers of σ (i. e., j is odd). These involve only sines of odd multiples. The reverse is the case in the $\bar{\theta}_{ijk}$ of S_3. If j is even, $\bar{\theta}_{ijk}$ involves only sines of odd multiples of τ. If j is odd, $\bar{\theta}_{ijk}$ involve only cosines of even multiples of τ.

Second. ρ_1 and ρ_2 are cosine series. The terms independent of λ involve only even multiples of τ.

σ_1 and σ_2 are sine series. The terms independent of λ involve only odd multiples of τ.

It is seen then that among the terms in R_3 which are independent of λ only cosines of even multiples of τ enter; and among the terms in S_3 which are independent of λ only sines of odd multiples enter. In the process of integration therefore two types of non-periodic terms arise, first, those coming from the terms which involve the first multiple of $\lambda\tau$, and second, those coming from the terms independent of λ. It is important therefore to separate the various terms into three classes:

(*a*) terms independent of λ,

(*b*) terms involving first multiple of $\lambda\tau$ only,

(*c*) terms involving multiples of $\lambda\tau$ higher than the first.

The solution for these last terms is periodic. We rewrite then the differential equations (199) as follows:

$$\rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = \epsilon_3\theta_{001} + \epsilon_2 f_1(\lambda,\tau) + f_2(\lambda,\tau) + f_3(\tau) + f_4(\kappa\lambda,\tau),$$

$$\sigma_{33}'' + \theta_4\sigma_3 + \theta_3\rho_3 = \qquad \epsilon_2 g_1(\lambda,\tau) + g_2(\lambda,\tau) + g_3(\tau) + g_4(\kappa\lambda,\tau).$$

The coefficients of ϵ_2 are, explicitly,

$$(200)\qquad \begin{aligned} f_1(\lambda,\tau) &= \theta_{101}\alpha_2 + 2\theta_{200}\alpha_2\alpha_6 + \theta_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\theta_{020}\gamma_1\gamma_6, \\ g_1(\lambda,\tau) &= \qquad 2\bar\theta_{200}\alpha_2\alpha_6 + \bar\theta_{110}(\alpha_2\gamma_6 + \gamma_1\alpha_6) + 2\bar\theta_{020}\gamma_1\gamma_6. \end{aligned}$$

These terms are homogeneous in the first degree in α_2 and γ_1, and consequently involve only terms which carry the first multiple of $\lambda\tau$. They are of importance since they carry the undetermined constant ϵ_2 as a factor. The solution for these terms has, by Theorem II, the form

$$\rho = F_1(\lambda,\tau) + b_1\tau\alpha_1(\tau), \qquad \sigma = G_1(\lambda,\tau) + b_1\tau\gamma_2(\tau),$$

where $F_1(\lambda, \tau)$ and $G_1(\lambda, \tau)$ are periodic and involve only terms carrying the first multiple of $\lambda\tau$. b_1 is a constant depending upon $f_1(\lambda,\tau)$ and $g_1(\lambda,\tau)$. It is found, by a calculation not difficult, to have the value

$$b_1 = 3 + \mu^2\wp(\mu^2).$$

The functions $f_2(\lambda,\tau)$ and $g_2(\lambda,\tau)$ have the same characters as $f_1(\lambda,\tau)$ and $g_1(\lambda,\tau)$. They are considered separately since they are independent of ϵ_2. Their solutions may be written

$$\rho = F_2(\lambda,\tau) + b_2\tau\alpha_1(\tau), \qquad \sigma = G_2(\lambda,\tau) + b_2\tau\gamma_2(\tau).$$

The terms $f_3(\tau)$ and $g_3(\tau)$ are independent of λ. $f_3(\tau)$ contains only cosines of even multiples of τ, while $g_3(\tau)$ contains only sines of odd multiples of τ. The solution for these terms has the form

$$\rho = F_3(\tau) + b_3\tau\alpha_3(\tau), \qquad \sigma = G_3(\tau) + b_3\tau\gamma_4(\tau).$$

Finally, $f_4(k\lambda,\tau)$ and $g_4(k\lambda,\tau)$ involve only terms which carry multiples of $\lambda\tau$ higher than the first. The solution for these terms is

$$\rho = F_4(k\lambda,\tau), \qquad \sigma = G_4(k\lambda,\tau).$$

The entire solution is then

$$
(201)\quad
\begin{aligned}
\rho_3 &= A^{(3)}\alpha_2(\tau) + D^{(3)}[\tau\alpha_3(\tau) + \alpha_4(\tau)] + \epsilon_3[\alpha_5(\tau) + a\tau\alpha_3(\tau)] \\
&\quad + \epsilon_2[F_1(\lambda, \tau) + b_1\tau\alpha_1(\tau)] + [F_2(\lambda, \tau) + b_2\tau\alpha_1(\tau)] \\
&\quad + [F_3(\tau) + b_3\tau\alpha_3(\tau)] + F_4(k\lambda, \tau), \\
\sigma_3 &= A^{(3)}\gamma_1(\tau) + D^{(3)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + \epsilon_3[\gamma_5(\tau) + a\tau\gamma_4(\tau)] \\
&\quad + \epsilon_2[G_1(\lambda, \tau) + b_1\tau\gamma_2(\tau)] + [G_2(\lambda, \tau) + b_2\tau\gamma_2(\tau)] \\
&\quad + [G_3(\tau) + b_3\tau\gamma_4(\tau)] + G_4(k\lambda, \tau).
\end{aligned}
$$

All of the functions $\alpha_i(\tau)$, $\gamma_i(\tau)$, $F_i(\tau)$, and $G_i(\tau)$ are periodic. In order that ρ_3 and σ_3 shall be periodic it is necessary and sufficient that the coefficient of $\tau\alpha_3(\tau)$ and $\tau\gamma_3(\tau)$, and the coefficient of $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$ be zero. That is,

$$D^{(3)} = -b_3 - a\epsilon_3,$$

and

$$\epsilon_2 = -\frac{b_2}{b_1},$$

by which condition the value of ϵ_2 is determined. In order to satisfy the initial conditions we must have $\rho_3 = 0$ at $\tau = 0$, and this determines $A^{(3)}$,

$$A^{(3)} = b_3\alpha_4(0) - \epsilon_3\alpha_6(0) + \frac{b_2}{b_1}F_1(0) - F_2(0) - F_3(0) - F_4(0).$$

Thus all the constants are determined except ϵ_3, and the solution is

$$
(202)\quad
\begin{aligned}
\rho_3 &= A^{(3)}\alpha_2(\tau) - b_3\alpha_4(\tau) + \epsilon_3\alpha_6(\tau) - \frac{b_2}{b_1}F_1(\lambda, \tau) + F_2(\lambda, \tau) + F_3(\tau) + F_4(k\lambda, \tau), \\
\sigma_3 &= A^{(3)}\gamma_1(\tau) - b_3\gamma_3(\tau) + \epsilon_3\gamma_6(\tau) - \frac{b_2}{b_1}G_1(\lambda, \tau) + G_2(\lambda, \tau) + G_3(\tau) + G_4(k\lambda, \tau).
\end{aligned}
$$

The constant ϵ_3 will be determined in satisfying the periodicity condition for the coefficient of e^4. This process of integration can be continued indefinitely. ρ_3 and σ_3 have the same properties that have been stated for ρ_2 and σ_2. It is evident from the properties of the differential equations that these properties persist for ρ_4 and σ_4, and so on indefinitely. The coefficient for ϵ_{i-1} in so far as it carries the first multiples of $\lambda\tau$ is always the same as for ϵ_2. Therefore the constant ϵ_{i-1} can always be determined so as to avoid non-periodic terms of the type $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$. The constant $D^{(i)}$ of integration can always be determined so as to destroy non-periodic terms of the type $\tau\alpha_3(\tau)$ and $\tau\gamma_4(\tau)$. The constant $A^{(i)}$ can always be determined so as to satisfy the initial conditions. The analysis of the type of terms entering is the same as for the subscript 3.

We have therefore a periodic solution with the period $2K\pi$ which is different from the class of generating orbits from which we set out, for the particle makes many revolutions before its orbit reënters. After integrating the equation $dv/d\tau = c/r^2$ the solution will contain five arbitrary constants (subject to the condition that λ must be rational), corresponding to mean distance, eccentricity, inclination, node, and epoch.

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